Eigenray tracing in 3D heterogeneous media using Spectral Element Method

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Summary

We present a new ray bending method, referred to as Eigenray tracing, for solving two-point boundary-value ray tracing problems in 3D heterogeneous velocity media. We apply a non-linear spectral element scheme to find the ray path between two endpoints, satisfying Fermat’s principle of least traveltime. In order to capture all arrivals between the endpoints, we suggest starting with the conventional iterative initial-value shooting method, and using the proposed method only for cases where it fails.

Introduction

Conventional two-point ray tracing in general 3D heterogeneous media is normally performed using the shooting method. In this method, a fan of rays is traced from a given starting point to the acquisition surface, where groups of rays arriving in the vicinity of each target location (e.g., a receiver) with similar take-off angles, is used for the convergence. By covering a wide range of take-off angles, multi-pathing stationary solutions can be found. However, in complex geological areas, the solution to the initial-value problem associated with the fan of rays is highly sensitive to small changes in the take-off angles, resulting in internal and external shadow zones which the numerical rays cannot penetrate. The proposed Eigenray method attempts to fill in the “missing” rays using the following workflow: a) constructing initial-guess rays by interpolating/extrapolating from the nearby rays, and b) updating the initial-guess rays by directly solving Fermat’s principle of stationary traveltime. This method is particularly attractive for areas that involve sharp velocity variations or local velocity anomalies. The case of simulating head waves is an extreme example of Eigenrays providing stable solutions where conventional ray tracing fails.

Ray bending methods have been extensively studied. Westwood and Vidmar (1987) applied the method to simulate the signals interacting with a layered ocean bottom. Waltham (1988) studied models consisting of constant-velocity layers separated by curved interfaces and computed ray paths, whose traveltimes are stationary with respect to changes in ray/interface intersection positions. The amplitude of the resulting event is related to the second derivative of the traveltime with respect to changes in the position of the ray/interface intersection. Moser (1991) used this method to compute the traveltime between the source point and all points of a network. Moser et al. (1992) improved the conventional ray bending approach by applying a) gradient search methods and b) interpolation by beta-splines between the nodes. Farra (1992) applied the Hamiltonian formulation to the ray bending approach. Shashidhar and Anand (1995) solved the problem of three-dimensional Eigenray tracing in an ocean channel. Bona et al. (2009) demonstrated that Fermat principle of stationary traveltime holds for general heterogeneous anisotropic media using a stochastic simulated annealing global search method. Recently, Sripanich and Fomel (2014) presented an efficient algorithm for two-point ray tracing in layered media by means of the bending method, where the ray paths are discretized at the intersection of the rays with the structure’s interfaces. In this study, we demonstrate the power of applying this type of ray tracing by using a non-linear spectral element method to efficiently find accurate least-time ray paths in complex geological areas, providing solutions beyond conventional ray tracing.

Model and Method

Consider an initial-guess ray trajectory that is divided into a number of multi-nodal segments (spectral elements) with a Legendre polynomial interpolation. Each segment includes two endpoints and a number of internal points placed along the internal flow parameter sampled according to the Gauss-Lobatto quadrature. The quadrature also associates definite weights for all points of a segment. The simplest spectral element is a three-nodal segment which we use to demonstrate the synthetic examples. The nodes are located at \(-1, 0\) and \(+1\) of the flow parameter. For each segment, we compute the local traveltimes and its first and second derivatives with respect to the nodal locations. The local derivatives are combined into the global derivatives of the traveltimes of the entire path. The local derivatives are vectors of size \(3n\), and the local second derivatives are matrices of the same size, where \(n\) is the number of segment nodes, including its endpoints. The global matrix of the second derivatives has a narrow band structure of width \(3n\). This matrix is symmetric, and if the stationary traveltime is minimum (the most practical case), it is also positive-definite in the proximity of the stationary ray. For the least-time solution, the first derivatives vanish. Knowledge of the second derivatives makes it possible to apply the Newton method for the Eigenray optimization. The second derivatives of the traveltime depend on the locations of the nodes; thus, the minimization problem is nonlinear. In addition, the second derivatives of the traveltime may be used to compute the geometrical spreading when the stationary ray path is found. We note that the least traveltime (or generally, the stationary traveltime) fully defines the ray path, but it still allows
some freedom in moving the nodes along the ray. We therefore apply an additional constraint on the lengths between the successive nodes, such that the nodes are located more densely in regions where the velocity changes rapidly. Thus, the constraints are the ratios between the arc lengths connecting successive nodes. There are, in general, two ways to implement the constraints: stiff constraints (e.g., applying Lagrangian multipliers method) and soft or relaxed constraints, by adding a penalty term to the target function to be minimized. The soft constraint method is simpler, does not lead to additional unknown parameters, and does not increase the bandwidth of the resolving matrix, while still providing excellent accuracy.

**Synthetic Examples**

Three examples of Eigenray solutions are presented. **Example 1:** High-velocity half-space under constant velocity layer (head wave). Consider a 1D velocity model whose vertical profile and gradient are shown in Figure 1a. The smoothed velocity model is depth dependent and described by,

\[ v = v_0 + \frac{\Delta v}{2} \left( 1 + \tanh \frac{z-z_0}{z_0} \right) \]

where \( v_0 = 2 \text{km/s} \) is the velocity of the “homogeneous” layer above the half-space, \( \Delta v = 2 \text{km/s} \) is the difference between the velocity of the half-space and that of the overlying layer; thus, the half-space velocity is \( v_h = v_0 + \Delta v = 4 \text{km/s} \). Actually, neither the overlying layer nor the half-space is homogeneous due to the transition zone. Parameter \( z_0 = 1.5 \text{km} \) is the mid-level of the transition zone, and parameter \( z_c = 0.2 \text{km} \) is the characteristic distance that shows the width of the transition zone. The offset \( h = 10 \text{ km} \). Figure 1b shows the initial guesses and the stationary ray. We applied the initial guess twice: first with a floor depth of 2 km (which is approximately the lower end of the transition zone), then with an over-estimated floor depth of 5 km (red and blue lines, respectively). Both initial guesses converge to a unique head wave solution (black line). In the next example we show that in the case of multi-arrivals, the final solution is sensitive to the initial guess.

**Example 2:** One-way path in a medium with low-velocity elliptic anomaly. Consider a constant background velocity with an elliptic anomaly region of a lower velocity, as shown in Figure 2. Location of the ellipse center is \( x_c, z_c \), and the semi-axes of the ellipse are \( r_x, r_z \). The velocity field is described by an analytic function,

\[ v(x, z) = v_b + \frac{\Delta v}{2} \left( \tanh A - 1 \right) \]

where \( v_b \) is the background velocity outside the ellipse, and \( v_b - \Delta v \) is the anomalous low velocity inside the ellipse. Negative \( \Delta v \) leads to anomalous high velocity inside the ellipse. Parameter \( s \) is the smoothing scale: the smaller \( s \) is, the sharper the velocity change. For infinitesimal \( s \), the velocity function becomes discontinuous. We accept the following parameters,
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\[ v_h = 5 \text{km/s}, \Delta v = 3 \text{km/s}, \ x_c = 5 \text{km}, \ z_c = 3 \text{km}, \ r_x = 3 \text{km}, \ r_z = 2 \text{km}, \ s = 0.2. \]  \hfill (4)

The source is located at the subsurface point with zero horizontal coordinate and depth \( z_d = 6 \text{km} \), and the receiver is on the surface, with the one-way offset \( s_d = 10 \text{km} \). Three different initial guesses, shown in Figure 2a by green, black and blue lines, lead to three different solutions shown in Figure 2b. The red ellipse is the contour of the velocity anomaly.

Example 3: Velocity field with two elliptic anomalies. Consider a model that combines slow- and high-velocity anomalies and fast half-space. It can be analytically described by,

\[ v(x,z) = v_h - \frac{\Delta v}{2} \left(1 - \tanh A_1\right) + \frac{\Delta v}{2} \left(1 - \tanh A_2\right) + \frac{\Delta v_h}{2} \left(1 + \tanh A_3\right), \]  \hfill (5)

where

\[ A_1 = \frac{1}{s} \left(\frac{(x-x_1)^2}{r_x^2} + \frac{(z-z_c)^2}{r_z^2} - 1\right), \]

\[ A_2 = \frac{1}{s} \left(\frac{(x-x_2)^2}{r_x^2} + \frac{(z-z_c)^2}{r_z^2} - 1\right), \]

\[ A_3 = \frac{z-z_0}{z_d}. \]  \hfill (6)

Parameters \( x_1 \) and \( x_2 \) are horizontal coordinates of central points of elliptic anomalies, \( z_c \) is their common vertical coordinate, \( z_0 \) is the floor depth of the high-velocity half-space. Parameters \( s \) and \( z_d \) govern the width of the transition zones,

\[ v_h = 4 \text{km/s}, \Delta v = 2 \text{km/s}, \ \Delta v_h = 3 \text{km/s}, \ x_1 = 4 \text{km}, \ x_2 = 20 \text{km}, \ z_c = 2 \text{km}, \ r_x = 4 \text{km}, \ r_z = 1 \text{km}, \ z_0 = 4 \text{km}, \ z_d = 0.2 \text{km}, \ s = 0.2, \]  \hfill (7)

and the offset \( h = 22 \text{km} \).
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Figures 3a and 3b show the velocity field and the ray tracing results, respectively. The distances in the velocity field plot are in km, and velocity is in km/s. In Figure 3b, the blue and red ellipses show the low-velocity and high-velocity anomalies, respectively. The gray line is the location of the half-space floor midline. The green line is the initial guess, where the depth of the floor was deliberately overestimated, in order to demonstrate convergence to a correct result even with an initial error. The black line is the least-time ray path.

Discussion on Eigenray Tracing in Anisotropic Medium

The spectral element analysis can be applied for Eigenray tracing in general anisotropic velocity media. With the Legendre polynomials we can interpolate not only for the location of the trajectory points between the nodes, but also for the Cartesian components of the ray path direction. The tangent to the path represents group (ray) velocity direction. Given the medium stiffness tensor, we can compute the ray velocity magnitude and its derivatives with respect to the nodal locations of the finite element. This will require a numerical procedure, because there is no explicit function that relates the group velocity to the group direction. The Christoffel equation relates the phase direction with the phase velocity and polarization vector, which, in turn, define the group velocity direction and magnitude.

A more robust approach for anisotropic ray tracing is to apply Hermitian finite elements with their corresponding interpolation, rather than Lagrange or Legendre. For each node of the Hermitian element, the degrees of freedom include the Cartesian location components and their derivatives with respect to the arc length. The location and direction of the ray trajectory between the element nodes is defined by both locations of the nodes and ray directions at the nodes. The ray direction in this case is represented by additional independent degrees of freedom (DOF) (which may be advantageous for anisotropic tracing). In 3D space each node has three location components and three direction components, but since the direction is described by a tangent vector of unit length, the number of independent DOF is five per node. The simplest two-nodal Hermitian element has ten DOF.

Conclusions

A ray bending algorithm, referred to as the Eigenray method, has been developed for two-point tracing in general 3D heterogeneous media. The method uses spectral elements with Legendre polynomial interpolation between the nodes and Gauss-Lobatto quadrature. Additional soft constraints related to the segment lengths govern the locations of the nodes along the stationary ray. Explicit expressions for the traveltime and its first and second derivatives allow the implementation of the Newton method of optimization, which also yields the geometrical spreading. We demonstrated the attractiveness of the method in cases of local low/high velocity anomalies and for solving the ray trajectory of head waves, which is beyond conventional ray tracing capabilities. For multi-arrivals, the final solution is obviously sensitive to the initial guess. We suggest a ray tracing strategy, where we start with a global ray shooting method to find all possible arrivals, and then use the proposed Eigenray method to fill in all missing rays within the internal and external shadow zones.
REFERENCES


