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Fourth-order normal moveout velocity in elastic layered orthorhombic media — Part 2: Offset-azimuth domain

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ABSTRACT

Based on the theory derived in part 1, in which we obtained the azimuthally dependent fourth-order normal-moveout (NMO) velocity functions for layered orthorhombic media in the slowness-azimuth/slowness and the slowness-azimuth/offset domains, in part 2, we extend the theory to the offset-azimuth/slowness and offset-azimuth/offset domains. We reemphasize that this paper does not suggest a new nonhyperbolic traveltime approximation; rather, it provides exact expressions of the NMO series coefficients, computed for normal-incidence rays, which can then be further used within known azimuthally dependent traveltime approximations for short to moderate offsets. The same type of models as in part 1 are considered, in which the layers share a common horizontal plane of symmetry, but the azimuths of their vertical symmetry planes are different. The same eight local (single-layer) and global (overburden multilayer) effective parameters are used. In addition, we have developed an alternative set of global effective parameters in which the "anisotropic" effective parameters are normalized, classified into two groups: two "azimuthally isotropic" parameters and six "azimuthally anisotropic" parameters. These parameters have a clearer physical interpretation and they are suitable for inversion purposes because they can be controlled and constrained. Next, we propose a special case, referred to as "weak azimuthal anisotropy," in which only the azimuthally anisotropic effective parameters are assumed to be weak. The resulting NMO velocity functions are considerably simplified, reduced to the form of the slowness-azimuth/slowness formula. We verify the correctness of our method by applying it to a multilayer orthorhombic medium with strong anisotropy. We introduce our derived, fourth-order slowness-azimuth/offset domain NMO velocity function into the well-known nonhyperbolic asymptotic traveltime approximation, and we compare the approximate traveltimes with exact traveltimes obtained by two-point ray tracing. The comparison shows an accurate match up to moderate offsets. Although the accuracy with the weak azimuthal anisotropic formula is inferior, it can still be considered reasonable for practical use.

INTRODUCTION

In part 1 of this paper, we reviewed the existing studies on azimuthally dependent nonhyperbolic traveltime approximations in homogeneous and multilayer orthorhombic media. In part 2, we only mention a few studies that we consider to be the most relevant to this part. Al-Dajani et al. (1998) study the azimuthally dependent quartic coefficient function for homogeneous media and conclude that for general anisotropy, even a single layer is characterized by five independent parameters:

$$A_4(\psi) = A_4^{(1)} \sin^4 \psi + A_4^{(2)} \cos^4 \psi + A_4^{(x)} \sin^2 \psi \cos^2 \psi + A_4^{(x2)} \sin \psi \cos^3 \psi + A_4^{(x3)} \sin^3 \psi \cos \psi, \quad (1)$$

where, for homogeneous media, the azimuth ψ can be assigned to the azimuth of the source-receiver offsets (offset azimuth) ψ_{off} or the azimuth of the ray (group) velocity ψ_{ray}. According to Al-Dajani and Toksoz (2001), for a homogeneous monoclinic layer, the components of the quartic coefficient can be obtained from the vertical-slowness surface $p_3 = q(p_1, p_2)$ and its partial derivatives with respect to the horizontal-slowness components p_1, p_2 (Grechka et al., 1999). In the case of homogeneous orthorhombic media, due to the orthogonal vertical symmetry planes, the last two components of the quartic coefficient $A_4^{(x2)}$ and $A_4^{(x3)}$ vanish. However, $A_4^{(1)}, A_4^{(2)}$, and $A_4^{(x)}$ are presented in a form that does not allow extension of the results to a multilayer model. We will later show that this result can be obtained as a particular

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case of our more general multilayer solution. Therefore, Al-Dajani and Toksoz (2001) suggest an approximation of $A_4(\psi_{\text{off}})$ in multilayer orthorhombic media by applying for each given azimuth the averaging formula for transverse isotropy with vertical symmetry axis (VTI). A similar averaging approach is suggested by Tsvankin (2005) and Tsvankin and Grechka (2011). Stovas (2015) studies the fourth-order normal moveout in acoustic multilayer orthorhombic media in the slowness- and offset-azimuth domains, using the vertical-slowness surface and its derivatives for each individual layer. He suggests approximating the effective azimuthally dependent quartic coefficient with three fourth-order parameters (rather than five), applying a least-squares fit. In this case, the resulting effective medium is still considered orthorhombic with the attempt to account for the different azimuths of the vertical symmetry planes at each layer, with a single effective azimuth.

Our study can be considered an extension of the above-mentioned works for general elastic anisotropic layered media, avoiding weak anisotropy assumptions or acoustic approximations for P-waves. It is based on results obtained in part 1 of this paper, in which we derived a special set of eight local effective parameters for individual orthorhombic layers. As mentioned, eight local effective parameters are needed to represent the kinematics of near-vertical rays in any general anisotropic layer to fourth-order accuracy. In a recent work, Koren and Ravve (2016) extend the derivation of the fourth-order local effective parameters to general anisotropy (including tilted orthorhombic [TORT] and triclinic). Once the local effective parameters are derived, the computation of the corresponding global effective parameters becomes a straightforward Dix-type summation. In part 1, we derived the exact formula for the slowness-azimuth domain global fourth-order NMO velocity function. In part 2, we convert the slowness-azimuth domain fourth-order NMO velocity formula to obtain the offset-azimuth domain fourth-order NMO velocity formula. Thus, the resulting global fourth-order NMO velocity functions in both azimuth domains are valid for general vertically varying anisotropic layered media and for all types of wave modes, provided the reciprocity condition holds (the traveltime is considered as an even function of the offset). For monoclinic layered media with a common horizontal symmetry plane (whose particular case is orthorhombic), all pure-mode and converted waves satisfy this condition. For layered models characterized with tilted symmetries, such as transverse isotropy with tilted axis of symmetry (TTI), tilted orthorhombic, and general triclinic layered media, the reciprocity condition holds only for pure-mode waves.

We suggest an alternative normalized set of effective parameters that distinguish two azimuthally isotropic parameters and six azimuthally anisotropic parameters. The advantage of this new parameterization is that the normalized azimuthally anisotropic effective parameters can be controlled, providing an intuitive physical interpretation of the strength of the azimuthal anisotropic effect. We then investigate the case in which only the azimuthally anisotropic effective parameters are weak, referred to as the weak azimuthal anisotropy approximation. In such a case, all azimuthal domains (slowness-azimuth/slowness, slowness-azimuth/offset, offset-azimuth/slowness, and offset-azimuth/offset) coincide and the resulting formulas are reduced to the simple form of the slowness-azimuth/slowness second- and fourth-order NMO velocity formulas.

We test the correctness of our method using a single-layer and multilayer models with strong orthorhombic anisotropy and with varying azimuths of the mutually orthogonal vertical symmetry planes. We introduce our derived fourth-order NMO velocity functions (and hence the quartic coefficient and the effective anellipticity functions) into the well-known, nonhyperbolic asymptotic traveltime approximation (Vasconcelos and Tsvankin, 2006), and we compare the approximate traveltimes with the exact traveltimes obtained by two-point ray tracing. The accurate match obtained for moderate and even relatively large offsets gives us great confidence in our derived formula.

As in part 1, the detailed derivations have been moved to the appendices. In Appendix A, we derive the power series expansion for the azimuthal lag between the two azimuthal domains $\Delta\Psi = \psi_{\text{off}} - \psi_{\text{slw}}$, and the relationship between the offset azimuth ψ_{off} and slowness azimuth ψ_{slw}, for near-vertical rays (applying the horizontal slowness p_h as a small parameter). In Appendix B, we invert this power series from the slowness-azimuth domain to the offset-azimuth domain; i.e., we convert $\psi_{\text{off}}(\psi_{\text{slw}})$ into $\psi_{\text{slw}}(\psi_{\text{off}})$. In Appendices C and D, we use this result to derive the fourth-order NMO velocity in the offset-azimuth/slowness and offset-azimuth/offset domains, respectively, which is the main goal of this part of the paper. In Appendix E, we suggest an alternative set of global effective parameters, in which the effective parameters are normalized (unitless) and classified into two main groups: azimuthally isotropic and azimuthally anisotropic parameters. We then emphasize their clearer physical interpretation and their advantages for inversion purposes. In Appendix F, we further study the case in which the azimuthally anisotropic effective parameters are weak, referred to as weak azimuthal anisotropy. This results in simple approximate relationships for the azimuthally dependent fourth-order NMO velocity $V_4(\psi)$ and the effective anellipticity $\eta_{\text{eff}}(\psi)$. Next, in Appendix G, we consider the case of a single layer in which all anisotropic parameters are weak, and our general formula reduces to the well-known weak anisotropy formula. Finally, in Appendix H, we consider a single elastic orthorhombic layer with any strength of anisotropy, and we demonstrate that the quartic coefficient formula obtained by Al-Dajani et al. (1998) for P-waves is a particular case of the more general case derived in this paper for multilayer models. In this work, we use the anisotropic parameterization suggested for orthorhombic media by Tsvankin (1997), which is an extension of Thomsen (1986) parameterization for transverse isotropy. A list of notations is included at the end of the paper.

METHOD

Following the definitions for the second-order NMO velocity $V_2(\psi)$ and the quartic coefficient $A_4(\psi)$,

$$V_2^2(\psi) = \lim_{p_h \to 0} \frac{h_R^2(\psi) + h_T^2(\psi)}{t^2(\psi) - t_o^2},$$

$$A_4(\psi) = V_2^2 t_o^2 \cdot \lim_{p_h \to 0} \frac{[t^2(\psi) - t_o^2] V_2^2(\psi) - [h_R^2(\psi) + h_T^2(\psi)]}{[h_R^2(\psi) + h_T^2(\psi)]^2}, \tag{2}$$

proposed in equation 5 of part 1, the primary aim of part 2 is to obtain explicit expressions for V_2 (a well-known formula) and A_4 in the offset-azimuth/offset domain $\psi = \psi_{\text{off}}$. We will then obtain the offset-azimuth/offset domain fourth-order NMO velocity V_4 and the effective anellipticity $\eta_{\rm eff}$:

$$V_4^4(\psi_{\rm off}) = V_2^4(\psi_{\rm off})[1-4A_4(\psi_{\rm off})],$$
$$\eta_{\rm eff}(\psi_{\rm off}) = \frac{V_4^4(\psi_{\rm off}) - V_2^4(\psi_{\rm off})}{8V_2^4(\psi_{\rm off})} = -\frac{A_4(\psi_{\rm off})}{2}. \quad (3)$$

In part 1, we introduced a natural coordinate system in which we defined the radial offset component h_R, in the direction of the slowness azimuth (phase-velocity azimuth), and the transverse offset component h_T, perpendicular to the radial direction. In equation 15 of part 1, we obtained the third-order power series for the radial $h_R(\psi_{\rm slw}, p_h)$ and transverse $h_T(\psi_{\rm slw}, p_h)$ components of the offset and the fourth-order series for the two-way traveltime $t(\psi_{\rm slw}, p_h)$ in the slowness-azimuth/slowness domain:

$$\begin{aligned}
h_R(\psi_{\rm slw}, p_h) =& (U_2 + W_{2x}\cos 2\psi_{\rm slw} + W_{2y}\sin 2\psi_{\rm slw})p_h \\
&+ (U_4 + W_{42x}\cos 2\psi_{\rm slw} + W_{42y}\sin 2\psi_{\rm slw} \\
&+ W_{44x}\cos 4\psi_{\rm slw} + W_{44y}\sin 4\psi_{\rm slw})p_h^3 + O(p_h^5), \\
h_T(\psi_{\rm slw}, p_h) =& -(W_{2x}\sin 2\psi_{\rm slw} - W_{2y}\cos 2\psi_{\rm slw})p_h \\
&- \left(\frac{1}{2}W_{42x}\sin 2\psi_{\rm slw} - \frac{1}{2}W_{42y}\cos 2\psi_{\rm slw}\right. \\
&\left. + W_{44x}\sin 4\psi_{\rm slw} - W_{44y}\cos 4\psi_{\rm slw}\right)p_h^3 + O(p_h^5), \\
t(\psi_{\rm slw}, p_h) =& t_{\rm o} + \frac{1}{2}(U_2 + W_{2x}\cos 2\psi_{\rm slw} + W_{2y}\sin 2\psi_{\rm slw})p_h^2 \\
&+ \frac{3}{4}(U_4 + W_{42x}\cos 2\psi_{\rm slw} + W_{42y}\sin 2\psi_{\rm slw} \\
&+ W_{44x}\cos 4\psi_{\rm slw} + W_{44y}\sin 4\psi_{\rm slw})p_h^4 + O(p_h^6),
\end{aligned} \quad (4)$$

where the small parameter in the series is the horizontal slowness p_h. In addition to the zero-offset time $t_{\rm o}$, the series includes eight constant coefficients: three second-order parameters U_2, W_{2x}, W_{2y} and five fourth-order parameters $U_4, W_{42x}, W_{42y}, W_{44x}, W_{44y}$. Note that we use the terms second- and fourth-order effective parameters, although the orders only refer to the traveltime series because the offset components are described with first- and third-order terms in the horizontal slowness. Traveltime is an even function of horizontal slowness, whereas both offset components are odd functions. Comparing the two offset components for a multilayer effective model, we compute the global azimuthal lag between the offset azimuth and slowness azimuth, $\Delta\Psi = \psi_{\rm off} - \psi_{\rm slw}$. Because both offset components have been obtained in part 1 in the slowness-azimuth domain, the azimuthal lag is presented as a function of the slowness azimuth as well. This leads to a power series expansion $\psi_{\rm off}(\psi_{\rm slw})$, with the same small parameter p_h. This power series is then inverted, resulting in a series $\psi_{\rm slw}(\psi_{\rm off})$, which can be introduced into the offset components and traveltime in the slowness-azimuth/slowness domain, to obtain these values in the offset-azimuth/slowness domain:

$$\begin{aligned}
h_R(\psi_{\rm off}, p_h) &= h_R[\psi_{\rm slw}(\psi_{\rm off}), p_h], \\
h_T(\psi_{\rm off}, p_h) &= h_T[\psi_{\rm slw}(\psi_{\rm off}), p_h], \\
t(\psi_{\rm off}, p_h) &= t[\psi_{\rm slw}(\psi_{\rm off}), p_h].
\end{aligned} \quad (5)$$

Once equation set 5 is obtained, equation set 4 yields the second- and fourth-order NMO velocities in the offset-azimuth/slowness domain. Next, $h_R(\psi_{\rm off}, p_h)$, $h_T(\psi_{\rm off}, p_h)$, and $t(\psi_{\rm off}, p_h)$ are introduced into the definition equation 2, to obtain the explicit relations for the offset-azimuth/offset domain second-order NMO velocity function (a well-known formula) and the newly derived, quartic coefficient function. Using equation 3, we further obtain the corresponding azimuthally dependent fourth-order NMO velocity and the effective anellipticity functions in the offset-azimuth/offset domain.

The three fourth-order effective parameters $A_4(\psi)$, $V_4(\psi)$, and $\eta_{\rm eff}(\psi)$ are closely related and equally describe the kinematics of near-vertical waves to fourth-order accuracy. Throughout the paper, these parameters with their different terms are sometimes used interchangeably, giving priority to their simplistic forms within the derivation process. In this paper, the simplest expression for the exact azimuthally dependent fourth-order effective parameter (coefficient) is obtained for the fourth-order NMO velocity.

OFFSET AZIMUTH VERSUS SLOWNESS AZIMUTH FOR NEAR-VERTICAL RAYS

The relationship between the transverse- and radial-offset components makes it possible to establish the azimuthal lag between the offset azimuth and the slowness azimuth:

$$\cos(\psi_{\rm off} - \psi_{\rm slw}) = \frac{h_R}{\sqrt{h_R^2 + h_T^2}},$$
$$\sin(\psi_{\rm off} - \psi_{\rm slw}) = \frac{h_T}{\sqrt{h_R^2 + h_T^2}}, \quad (6)$$

where h_R and h_T are the radial and transverse components of the offset, respectively, given in equation set 4. Equation 6, in turn, yields the series expansion for the offset azimuth versus slowness azimuth (see Appendix A for derivation details):

$$\cos\ \psi_{\rm off} = \cos\ \psi_{0,\rm off} - B_{\rm slw}\ \sin\ \psi_{0,\rm off}\,p_h^2 + O(p_h^4),$$
$$\sin\ \psi_{\rm off} = \sin\ \psi_{0,\rm off} + B_{\rm slw}\ \cos\ \psi_{0,\rm off}\,p_h^2 + O(p_h^4), \quad (7)$$

where the zero-offset terms of the offset azimuth are

$$\cos\ \psi_{0,\rm off} = \frac{(U_2 + W_{2x})\cos\ \psi_{\rm slw} + W_{2y}\ \sin\ \psi_{\rm slw}}{\sqrt{U_2^2 + W_2^2 + 2U_2W_2\ \cos\ 2(\psi_{\rm phs} - \Psi_{2,\rm H})}},$$
$$\sin\psi_{0,\rm off} = \frac{(U_2 - W_{2x})\sin\ \psi_{\rm slw} + W_{2y}\ \cos\ \psi_{\rm slw}}{\sqrt{U_2^2 + W_2^2 + 2U_2W_2\ \cos\ 2(\psi_{\rm slw} - \Psi_{2,\rm H})}}, \quad (8)$$

where $\Psi_{2,\rm H}$ is the azimuth of the high second-order NMO velocity and parameter $B_{\rm slw}$ is defined in Appendix A (equation A-3). Note that the third-order series for the offset components in equation 4 becomes a second-order series after introduction into equation 6, due to cancellation of the horizontal slowness p_h in the numerators and denominators on the right side. However, the second-order approximation of the offset azimuth is exactly what we need for the fourth-order approximation of the traveltime.

SLOWNESS AZIMUTH VERSUS OFFSET AZIMUTH FOR NEAR-VERTICAL RAYS

To compute the radial- and transverse-offset components and the traveltime in the offset-azimuth domain (equation 5), we need the series expansion for $\psi_{\mathrm{slw}}(\psi_{\mathrm{off}})$ rather than $\psi_{\mathrm{off}}(\psi_{\mathrm{slw}})$. For this, equation 7 has to be inverted (see Appendix B for derivation details). The result of the inversion reads

$$\begin{aligned}\cos\psi_{\mathrm{slw}} &= \cos\psi_{0,\mathrm{slw}} - B_{\mathrm{off}}\sin\psi_{0,\mathrm{slw}}\, p_h^2 + O(p_h^4),\\ \sin\psi_{\mathrm{slw}} &= \sin\psi_{0,\mathrm{slw}} + B_{\mathrm{off}}\cos\psi_{0,\mathrm{slw}}\, p_h^2 + O(p_h^4),\end{aligned} \tag{9}$$

where the zero-offset terms of the slowness azimuth are

$$\begin{aligned}\cos\psi_{0,\mathrm{phs}} &= \frac{(U_2 - W_{2x})\cos\psi_{\mathrm{off}} - W_{2y}\sin\psi_{\mathrm{off}}}{\sqrt{U_2^2 + W_{2x}^2 + W_{2y}^2 - 2U_2W_2\cos 2(\psi_{\mathrm{off}} - \Psi_{2,\mathrm{H}})}},\\ \sin\psi_{0,\mathrm{phs}} &= \frac{(U_2 + W_{2x})\sin\psi_{\mathrm{off}} - W_{2y}\cos\psi_{\mathrm{off}}}{\sqrt{U_2^2 + W_{2x}^2 + W_{2y}^2 - 2U_2W_2\cos 2(\psi_{\mathrm{off}} - \Psi_{2,\mathrm{H}})}},\end{aligned} \tag{10}$$

and parameter B_{off} is defined in equation B-11.

FOURTH-ORDER NMO VELOCITY IN THE OFFSET-AZIMUTH/SLOWNESS DOMAIN

Introduction of $\psi_{\mathrm{slw}}(\psi_{\mathrm{off}})$ from equation 9 into the traveltime equation of set 4 and taking into account that

$$t(\psi_{\mathrm{slw}}, p_h) = t_o + \frac{1}{2}V_2^2(\psi_{\mathrm{slw}})t_o p_h^2 + \frac{3}{8}V_4^4(\psi_{\mathrm{slw}})t_o p_h^4 + O(p_h^6), \tag{11}$$

make it possible to obtain the second- and fourth-order NMO velocities in the offset-azimuth/slowness domain. The fourth-order NMO velocity is presented in factorized form (see Appendix C for details),

$$V_{4,\mathrm{slw}}^4(\psi_{\mathrm{off}}) = K_{2,\mathrm{slw}}(\psi_{\mathrm{off}})K_{4,\mathrm{slw}}(\psi_{\mathrm{off}}), \tag{12}$$

where the normalizing factor $K_{2,\mathrm{slw}}(\psi_{\mathrm{off}})$ depends on the second-order global effective parameters only, whereas the fourth-order kernel $K_{4,\mathrm{slw}}(\psi_{\mathrm{off}})$ depends on the global effective second-order and fourth-order parameters. For each offset azimuth, the fourth-order kernel is a scalar value presented in bilinear form,

$$K_{4,\mathrm{slw}}(\psi_{\mathrm{off}}) = \mathbf{m}\mathbf{M}_{\mathrm{off/slw}}\mathbf{a}_{\mathrm{off},7}, \tag{13}$$

where $\mathbf{m}$ is a row vector that includes five fourth-order effective parameters (defined in equation A-5), $\mathbf{M}_{\mathrm{off/slw}}$ is a 5×7 matrix that depends solely on the second-order parameters (equations C-6 to C-12), and $\mathbf{a}_{\mathrm{off},7}$ is a column vector of length seven that depends on the offset azimuth alone (equation C-5).

FOURTH-ORDER NMO VELOCITY IN THE OFFSET-AZIMUTH/OFFSET DOMAIN

To obtain the second-order NMO velocity $V_{2,\mathrm{off}}(\psi_{\mathrm{off}})$ and the quartic moveout coefficient $A_4(\psi_{\mathrm{off}})$ in the offset-azimuth/offset domain, we apply the limits in equation 2. Equation 3 is then used to obtain the fourth-order velocity $V_{4,\mathrm{off}}(\psi_{\mathrm{off}})$ and the effective anellipticity$\eta_{\mathrm{eff}}(\psi_{\mathrm{off}})$. All three azimuthally dependent fourth-order effective parameters A_4, V_4, and η_{eff} can be equally used to describe traveltimes of azimuthally varying near-vertical rays. The fourth-order NMO velocity for the offset-azimuth domain is derived in Appendix C. Like in the case of the slowness-azimuth/offset and offset-azimuth/slowness domains, $V_{4,\mathrm{off}}(\psi_{\mathrm{slw}})$ and $V_{4,\mathrm{slw}}(\psi_{\mathrm{off}})$, the result can be presented in factorized form:

$$V_{4,\mathrm{off}}^4(\psi_{\mathrm{off}}) = K_{2,\mathrm{off}}(\psi_{\mathrm{off}})K_{4,\mathrm{off}}(\psi_{\mathrm{off}}), \tag{14}$$

where the normalizing factor $K_{2,\mathrm{off}}(\psi_{\mathrm{off}})$ depends on the second-order global effective parameters only, whereas the fourth-order kernel $K_{4,\mathrm{off}}(\psi_{\mathrm{off}})$ depends on the global effective second-order and fourth-order parameters. For each offset azimuth, the fourth-order kernel is a scalar value presented in bilinear form:

$$K_{4,\mathrm{off}}(\psi_{\mathrm{off}}) = \mathbf{m}\mathbf{M}_{\mathrm{off/off}}\mathbf{a}_{\mathrm{off},5}, \tag{15}$$

where $\mathbf{m}$ is a row vector that includes five fourth-order effective parameters (equation A-5), $\mathbf{M}_{\mathrm{off/off}}$ is a 5×5 matrix that depends solely on the second-order parameters (equations D-5 to D-7), and $\mathbf{a}_{\mathrm{off},5}$ is a column vector of length five that depends on the offset azimuth alone (equation B-11).

NORMALIZED SET OF GLOBAL EFFECTIVE PARAMETERS

The eight second- and fourth-order global effective parameters proposed in this work can be classified into two types of parameters: azimuthally isotropic $\{U_2, U_4\}$ and azimuthally anisotropic $\{W_{2x}, W_{2y}, W_{42x}, W_{42y}, W_{44x}, W_{44y}\}$. As mentioned, the main advantage of using these global effective parameters is the ability to compute them for any type of azimuthally anisotropic layered media in a straightforward Dix-type summation. However, especially for inversion purposes when many parameters are involved, it is recommended to work with normalized parameters, if possible. Normalized parameters can be well-constrained and have clearer physical interpretation. Moreover, when setting the normalized parameters to zero, the approximation formula is reduced in describing the simplest physical model. In Appendix D, we propose the following alternative set of "normalized" effective parameters: azimuthally isotropic (e.g., VTI type) effective parameters

$$\Big\{\underbrace{\bar{V}_2}_{\text{second-order}},\ \underbrace{\bar{\eta}_{\mathrm{eff}}}_{\text{fourth-order}}\Big\} \tag{16}$$

and azimuthally anisotropic effective parameters

$$\underbrace{\{\quad e_{2,}\Psi_{2,\mathrm{H}}}_{\text{second-order}},\underbrace{e_{4,\mathrm{L}},e_{4,\mathrm{H}},\Delta\Psi_{42},\Delta\Psi_{44}\}}_{\text{fourth-order}}. \tag{17}$$

Here, $\overline{V}_2$ is the only nonnormalized parameter, indicating the azimuthally isotropic second-order NMO velocity of the multilayer orthorhombic model, and $\bar{\eta}_{\mathrm{eff}}$ is the azimuthally isotropic effective anellipticity. The remaining six azimuthally anisotropic parameters include two second-order parameters and four fourth-order parameters. The second-order parameters are: e_2, the elliptic parameter, and $\Psi_{2,\mathrm{H}}$, the effective azimuth, indicating the direction of the high second-order NMO velocity. The fourth-order parameters are $e_{4,\mathrm{L}}$ and $e_{4,\mathrm{H}}$, the residual effective anellipticity parameters with respect to $\bar{\eta}_{\mathrm{eff}}$, associated with the azimuths of the low and high second-order NMO velocities, respectively, and two additional fourth-order residual azimuths $\Delta\Psi_{42}$ and $\Delta\Psi_{44}$. By decreasing the values of the three azimuthally anisotropic parameters e_2, $e_{4,\mathrm{L}}$, $e_{4,\mathrm{H}}$, the effective model approaches an azimuthally isotropic (VTI type) one. A detailed explanation and two-way relationships between the two sets of effective parameters are given in Appendix D. Both sets, original and normalized, are independent of the specific azimuth domain (slowness azimuth or offset azimuth). We still consider the original set of parameters to be important because it is used for internal calculations, and it is suitable for Dix-type operations. The normalized set of parameters can be used for inversion purposes and for communicating the actual physical meaning of the global effective parameters in a more intuitive manner.

PARTICULAR CASES

Multilayer weak azimuthal anisotropy

Consider a multilayer model in which at least some of the layers are characterized with, for example, the vertically fractured transverse isotropy (e.g., Schoenberg and Helbig [1997]; see also a brief review of this model in part 1: Numerical examples). Each layer can be described as a superposition of a background VTI and two orthogonal aligned vertical fracture planes. We assume that the strength of the VTI parameters can be of any order, whereas the vertical fractures are characterized by relatively small weaknesses. In Appendix E, we show that for this particular case, the azimuthally dependent fourth-order NMO velocity is essentially simplified:

$$\begin{aligned} V_2^2(\psi)t_\mathrm{o} &= U_2 + (W_{2x}\cos 2\psi + W_{2y}\sin 2\psi),\\ V_4^4(\psi)t_\mathrm{o} &= 2(U_4 + W_{42x}\cos 2\psi + W_{42y}\sin 2\psi\\ &\quad + W_{44x}\cos 4\psi + W_{44y}\sin 4\psi). \end{aligned} \tag{18}$$

In this case, the two azimuthally isotropic global effective parameters U_2 and U_4 are arbitrary, whereas the remaining azimuthally anisotropic parameters W_{2x}, W_{2y}, W_{42x}, W_{42y}, W_{44x}, and W_{44y} are considered small, allowing linearization. We call this approximation weak azimuthal anisotropy and we note that in this case, the equations for the second- and fourth-order NMO velocities are identical in all four domains (slowness-azimuth/slowness, slowness-azimuth/offset, offset-azimuth/slowness, and offset-azimuth/offset). Therefore, the azimuth ψ in equation 18 may stand for either ψ_{slw} or ψ_{off}. We further note that in this case, the azimuthal lag $\Delta\Psi = \psi_{\mathrm{off}} - \psi_{\mathrm{phs}}$ does not vanish and may be still essential (see equations F-7 and F-8).

The weak azimuthal anisotropy approximations for the second- and fourth-order NMO velocities (equation 18) and effective anellipticity (equation F-1), in terms of the normalized effective parameters, are listed in equation F-5.

Single-layer weak anisotropy

For a single layer, the global and local effective parameters coincide. Assuming weak orthorhombic parameters, the azimuthally dependent effective anellipticity is reduced to the formula obtained by Pech and Tsvankin (2004) (see Appendix F).

Single-layer strong anisotropy

In Appendix G, we consider compressional wave, and we compute the quartic coefficient in the offset-azimuth/offset domain for a single orthorhombic layer, without making any weak anisotropy assumptions, and we show that the well-known result obtained by Al-Dajani et al. (1998) is a particular case of our more general multilayer case.

NUMERICAL EXAMPLE FOR A SINGLE LAYER AND A PACKAGE OF LAYERS

The same asymptotic traveltime approximation for P-waves used in part 1 is also used in part 2, but in the offset-azimuth domain:

$$\frac{t^2(\psi_{\mathrm{off}},h) - t_\mathrm{o}^2}{t_\mathrm{o}^2} = \frac{h^2}{V_2^2(\psi_{\mathrm{off}})t_\mathrm{o}^2} - \frac{1}{V_2^2(\psi_{\mathrm{off}})t_\mathrm{o}^2} \cdot \frac{2\eta_{\mathrm{eff}}(\psi_{\mathrm{off}})h^4}{V_2^2(\psi_{\mathrm{off}})t_\mathrm{o}^2 + [1+2\eta_{\mathrm{eff}}(\psi_{\mathrm{off}})]h^2}. \tag{19}$$

As mentioned in part 1, equation 19 was initially suggested for VTI by Alkhalifah and Tsvankin (1995). Its validation for azimuthally anisotropic media was later justified by Al-Dajani et al. (1998), Xu et al. (2005), and Vasconcelos and Tsvankin (2006), invoking different approximations for $\eta_{\mathrm{eff}}(\psi_{\mathrm{off}})$. Our goal is to test the correctness of our new derived exact expression for $\eta_{\mathrm{eff}}(\psi_{\mathrm{off}})$.

We first test the accuracy of the nonhyperbolic moveout approximation in the offset-azimuth domain with a single-layer (homogeneous) orthorhombic model, whose properties are listed in Table 1 of part 1, and then with a multilayer model whose properties are listed in Tables 2 and 3 of part 1. In Figure 1, we plot the second- and fourth-order NMO velocities for the single orthorhombic layer and the effective anellipticity versus slowness azimuth and offset azimuth in all four domains. Note that for weak azimuthal anisotropy approximation, the NMO velocity functions in all domains converge to the slowness-azimuth/slowness domain. We can clearly see two vertical symmetry planes on these plots. At the vertical symmetry planes ψ_{x_1} and $\psi_{x_1}+\pi/2$, the second- and fourth-order NMO velocities in all azimuth domains are equal, $v_2(\psi_{\mathrm{slw}}) = v_2(\psi_{\mathrm{off}})$, $v_4(\psi_{\mathrm{slw}}) = v_4(\psi_{\mathrm{off}})$. In Figure 2, the NMO velocities and effective anellipticities for the multilayer model, Table 3 of part 1, are plotted in all azimuthal domains, along with their domain-independent, weak azimuthal anisotropy approximation, which converges to the slowness-azimuth/slowness domain. A similar identity holds for the global effective azimuths corresponding to the high and low second-order NMO velocities $\Psi_{2,\mathrm{H}}$, and $\Psi_{2,\mathrm{L}} = \Psi_{2,\mathrm{H}} + \pi/2$, $V_2(\psi_{\mathrm{slw}}) = V_2(\psi_{\mathrm{off}})$, $V_4(\psi_{\mathrm{slw}}) = V_4(\psi_{\mathrm{off}})$ in all domains. However, only the plots for the global second-order NMO velocity $V_2(\psi_{\mathrm{slw}})$ and $V_2(\psi_{\mathrm{off}})$ are symmetric because these plots are defined by a single azimuthal parameter: the azimuth of the high second-order NMO velocity $\Psi_{2,\mathrm{H}}$. The plots for the fourth-order NMO velocity are defined by $\Psi_{2,\mathrm{H}}$ and two additional global effective azimuths Ψ_{42} and Ψ_{44}, related to the fourth-order terms with the azimuth doubled and the azimuth quadrupled, respectively. Unlike homogeneous orthorhombic media, multilayer models with different azimuthal orientations of the orthorhombic symmetry planes at each layer have no common vertical planes of symmetry. Their traveltime-equivalent medium exhibits monoclinic characteristics rather than orthorhombic because the layers share only the horizontal plane of symmetry (and not the vertical ones). The NMO velocities in Figures 1 and 2 are normalized (divided by the average vertical velocity $V_{\mathrm{ave}} = 2z/t_{\mathrm{o}}$, where z is the depth of the reflector and t_{o} is the two-way vertical time). The gray vertical lines indicate the azimuths of the high and low second-order NMO velocities, and they are labeled accordingly. We emphasize that at these two azimuths, each of the three functions $V_2(\psi)$, $V_4(\psi)$, and $\eta_{\mathrm{eff}}(\psi)$ accepts identical values in all azimuthal domains.

In Figure 3, we plot the exact numerical traveltime $t = t(\psi_{\mathrm{off}}, h)$ and relative errors for the hyperbolic and nonhyperbolic approximations for the single-layer model versus offset magnitude and azimuth, comparing the approximations with the exact numerical solution. In Figure 3a–3c, the offset azimuth is constant, and the offset magnitude varies. The offset and the traveltime are normalized, $\bar{t} = t/t_{\mathrm{o}}$ and $\bar{h} = h/(2z)$, where z is the depth of the reflection point. The offset range is $0 \le \bar{h} \le 2.5$, and the azimuths considered are $\psi_{\mathrm{off}} = 0^\circ, 30^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ$. Due to the vertical symmetry planes, we see that the exact solution and the approximation errors for the complementary azimuths, ψ_{off} and $\pi - \psi_{\mathrm{off}}$, are identical; i.e., the plots for 30° and 150° or 60°and120° look alike. In Figure 3d–3f, we plot the traveltime and relative traveltime errors for fixed offsets $\bar{h} = 1, \ldots, 2.5$ with a step $\Delta\bar{h} = 0.25$, for continuously varying azimuth, $0 \le \psi_{\mathrm{off}} \le \pi$. Figure 3a and 3d shows numerical solutions, Figure 3b and 3e shows the relative errors of the hyperbolic approximation, and Figure 3c and 3f shows the relative errors of the nonhyperbolic approximation. In the worst case, the error for the hyperbolic approximation is 2.83%, and for the nonhyperbolic approximation, 0.324%. For our example, the maximum errors correspond to the azimuths of the vertical symmetry planes, and therefore these errors are identical in the slowness-azimuth/offset and offset-azimuth/offset domains. The contribution of the nonhyperbolic term is essential for obtaining the expected accurate results. It is interesting to note that for this particular model, the azimuthal dependency of the traveltime is relatively weak in the proximity of $\Psi_{2,\mathrm{L}}$.

As we see from Figure 3e and 3f, for offset azimuth $\psi_{\mathrm{off}} \approx \{43.8^\circ, 54.1^\circ\}$ (located on the interval $0 \le \psi_{\mathrm{off}} \le \pi/2$) and its symmetric azimuth $\psi_{\mathrm{off}} \approx \{133.8^\circ, 144.1^\circ\}$ (on the interval $\pi/2 \le \psi_{\mathrm{off}} \le \pi$), for each specific offset, the accuracy of the hyperbolic and nonhyperbolic approximations is equal because the effective anellipticity vanishes (and thus, the whole nonhyperbolic term as well). In Figure 4, to the left, we plot the accuracies of the hyperbolic and nonhyperbolic approximations, and we show the azimuths of the effective elliptic anisotropy (vertical gray lines). Figure 4b is a magnification of Figure 4a. The different colors in Figure 4a and 4b correspond to the different offsets, whose values are specified in the legend. Hyperbolic approximations are shown with the dashed lines, and nonhyperbolic approximations are shown with the solid lines. As we can see, at the azimuths of the effective elliptic anisotropy, each solid line intersects with its corresponding dashed line.

In Figure 5a, we plot the NMO velocity $v_2(\psi_{\mathrm{off}})$ and horizontal ray (group) velocity $v_{h,\mathrm{ray}}(\psi_{\mathrm{off}})$ versus the offset azimuth for the single-layer model. Figure 5b is a magnification of Figure 5a within the region of interest, $42^\circ \le \psi_{\mathrm{off}} \le 56^\circ$. We see that for $\psi_{\mathrm{off}} \approx 43.8^\circ$ and $\psi_{\mathrm{off}} \approx 54.5^\circ$, the NMO velocity and the horizontal ray velocity coincide, $v_2(\psi_{\mathrm{off}}) = v_{h,\mathrm{ray}}(\psi_{\mathrm{off}})$, which means that for these azimuths, the hyperbolic approximation also converges to the right asymptotic velocity for unbounded offsets. In Figure 1b, the graph of effective anellipticity versus offset azimuth shows that $\eta_{\mathrm{eff}}(\psi_{\mathrm{off}})$ vanishes for $\psi_{\mathrm{off}} \approx 43.8^\circ$, which is also

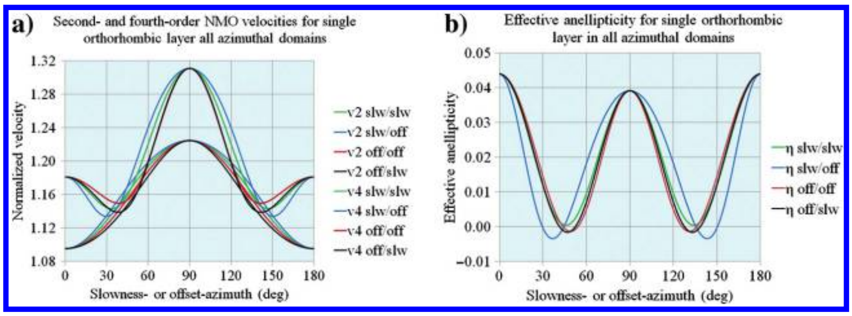

Figure 1. (a) Normalized second- and fourth-order local NMO velocities (v_2, v_4) and (b) local effective anellipticity $\eta_{\mathrm{eff}}^{\mathrm{loc}}$, for a single-layer orthorhombic model, in the four azimuthal domains. Domain labels: "slw/slw" — slowness-azimuth/slowness, "slw/off" — slowness-azimuth/offset, "off/slw" — offset-azimuth/slowness, and "off/off" — offset-azimuth/offset. Weak azimuthal anisotropy approximation in all domains converges to slowness-azimuth/slowness domain. The medium properties are listed in Table 1 of part 1.

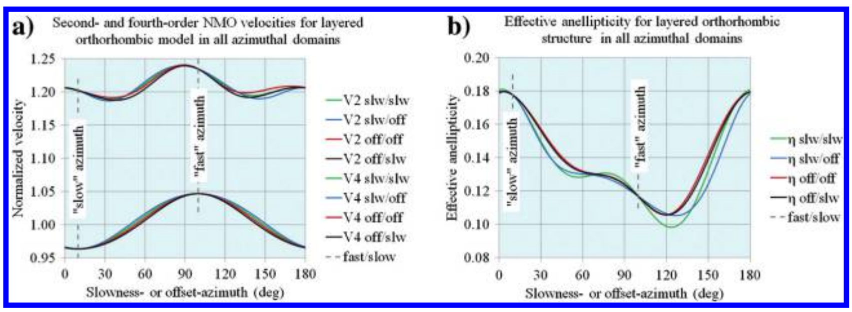

Figure 2. (a) Normalized second- and fourth-order global NMO velocities (V_2, V_4) and (b) global effective anellipticity η_{eff}, for a multilayer orthorhombic model (vertically fractured transverse isotropy), in the four azimuthal domains. Domain labels: slw/slw — slowness-azimuth/slowness, slw/off — slowness-azimuth/offset, off/slw — offset-azimuth/slowness, and off/off — offset-azimuth/offset. Weak azimuthal anisotropy approximation in all domains converges to slowness-azimuth/slowness domain. The medium properties are listed in Table 3 of part 1.

evidence for $v_2(\psi_{\mathrm{off}}) = v_4(\psi_{\mathrm{off}})$. Vanishing effective anellipticity is more likely the case in single-layer orthorhombic medium and not in a multilayer model. In a multilayer model, the effective anellipticity is normally positive for all azimuths due to an induced counterpart (yielded from different layer velocities).

In Figure 6, we analyze the accuracy of the hyperbolic and nonhyperbolic approximations for the multilayer model, Table 3 of part 1, applying an exact numerical ray tracing for comparison. In Figure 6a–6c, the azimuths are constant, $\psi_{\mathrm{off}} = 0^\circ, 30^\circ, 60^\circ, 90^\circ, 120^\circ$, and 150°, and the normalized offset varies continuously in the range $0 \le \bar{h} \le 2$. In Figure 6d–6f, the offset magnitudes are constant, $\bar{h} = 1, \ldots, 2$ with a step $\Delta\bar{h} = 0.25$, and the azimuth runs for all values, $0 \le \psi_{\mathrm{off}} \le \pi$. Figure 6a and 6d shows the exact ray-tracing solution, Figure 6b and 6e shows the relative errors of the hyperbolic approximation, and Figure 6c and 6f shows the relative errors of the nonhyperbolic approximation. For the given multilayer model, within the studied range of the offsets $\bar{h} \le 2$, and all azimuths, the maximum traveltime error of the hyperbolic approximation is approximately 11%, whereas that of the nonhyperbolic approximation is less than 1%. These results reemphasize the importance of using the fourth-order NMO velocity for analyzing azimuthally anisotropic velocity parameters.

DISCUSSION OF THE NUMBER OF INDEPENDENT HIGH-ORDER PARAMETERS

As mentioned above, a homogeneous (single-layer) orthorhombic model has only three independent fourth-order parameters, whereas a multilayer orthorhombic model has five independent fourth-order parameters, which may be parameterized, for example, by three coefficients with units velocity^4 × time (or three unitless effective anellipticities) and two additional fourth-order reference azimuths. Xu and Tsvankin (2006) and Wang and Tsvankin (2009) suggest adding only a single additional reference azimuth related to the effective anellipticity term of a multilayer orthorhombic model. Stovas (2015) suggests no additional azimuths, thus approximating the fourth-order kinematics of a multilayer orthorhombic medium by a single equivalent (effective) orthorhombic layer. For inversion problems, this may be sometimes justified because there is a tendency to keep the number of parameters at a minimum, even if it somewhat compromises the accuracy. However, our plots for the fourth-order velocity $V_4(\psi_{\mathrm{slw}}), V_4(\psi_{\mathrm{off}})$ and effective anellipticity $\eta_{\mathrm{eff}}(\psi_{\mathrm{slw}}), \eta_{\mathrm{eff}}(\psi_{\mathrm{off}})$ in Figure 2 demonstrate that these functions are more complex and a single additional azimuth does not suffice. As follows from equation 4, the hyperbolic (second order) terms of the moveout components have azimuthal

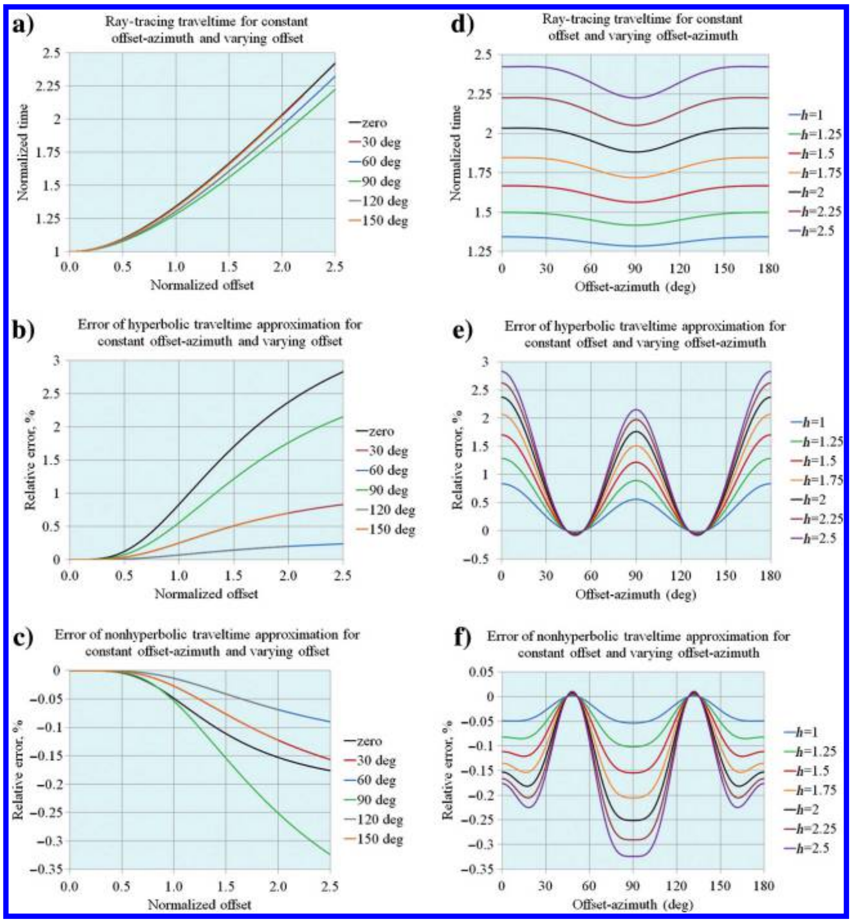

Figure 3. Relative error of hyperbolic and nonhyperbolic traveltime approximations versus offset and offset azimuth, for a single-layer orthorhombic model: (a) Ray-tracing traveltime versus offset, for constant offset azimuth; (b) relative error of hyperbolic traveltime approximation versus offset, for constant offset azimuth; (c) relative error of nonhyperbolic traveltime approximation versus offset, for constant offset azimuth; (d) ray-tracing traveltime versus offset azimuth, for constant offset; (e) relative error of hyperbolic traveltime approximation versus offset azimuth, for constant offset; and (f) relative error of nonhyperbolic traveltime approximation versus offset azimuth, for constant offset. The medium properties are listed in Table 1 of part 1.

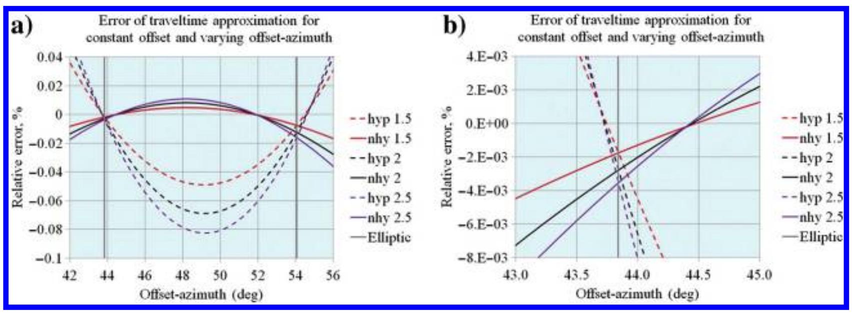

Figure 4. Error of traveltime approximation for constant offset and varying offset azimuth, for a single-layer orthorhombic model: (a) Azimuths of effective elliptic anisotropy; and (b) magnification for one of these azimuths. The medium properties are listed in Table 1 of part 1.

multiplicity $\omega_2 = 2$ and effective reference azimuth $\Psi_{2,H}$, whereas the nonhyperbolic (fourth order) terms have two azimuthal multiplicities (basic and doubled) $\omega_{42} = 2$ and $\omega_{44} = 4$ related to their corresponding, generally different, effective reference azimuths Ψ_{42} and Ψ_{44}.

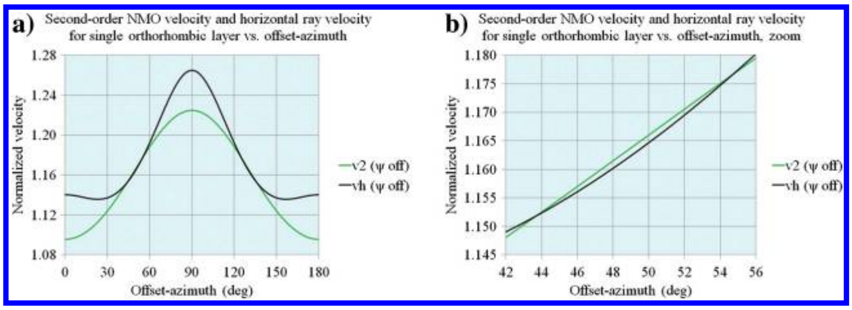

Figure 5. Second-order NMO velocity and horizontal ray velocity and their magnification (to the right), for a single-layer orthorhombic model: (a) Full-azimuth range; and (b) magnification for the region of interest. The medium properties are listed in Table 1 of Part 1.

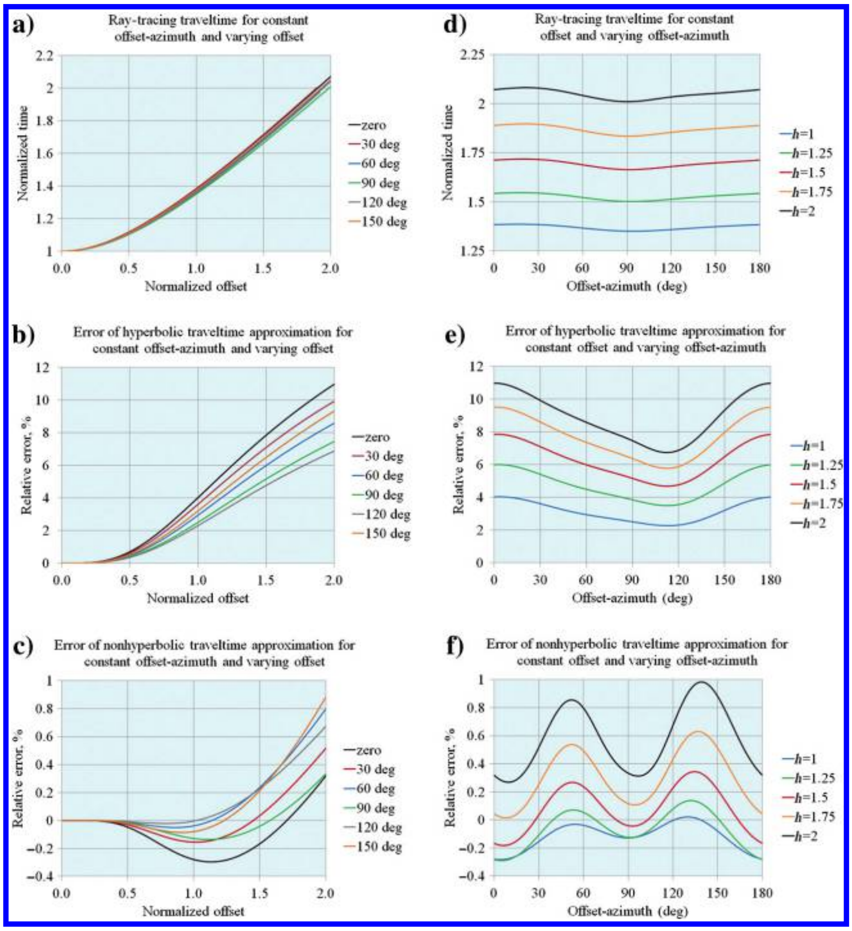

Figure 6. Relative error of hyperbolic and nonhyperbolic traveltime approximations versus offset and offset azimuth, for a multilayer orthorhombic model: (a) Ray-tracing traveltime versus offset, for constant offset azimuth; (b) relative error of hyperbolic traveltime approximation versus offset, for constant offset azimuth; (c) relative error of nonhyperbolic traveltime approximation versus offset, for constant offset azimuth; (d) ray-tracing traveltime versus offset azimuth, for constant offset; (e) relative error of hyperbolic traveltime approximation versus offset azimuth, for constant offset; and (f) relative error of nonhyperbolic traveltime approximation versus offset azimuth, for constant offset. The medium properties are listed in Table 3 of part 1.

Performing any synthetic exercise (with a multilayer orthorhombic model) that includes first forward and then inverse Dix-like transforms, we obtain three fourth-order local effective parameters for each layer. However, when using this method to invert real field data, the number of independent local high-order parameters should be the same as the number of global fourth-order parameters — five. The main reason is that in reality, the inverted layer is never purely orthorhombic. It may be, for example, monoclinic with only a horizontal plane of symmetry, and five independent high-order coefficients are required (Al-Dajani et al., 1998; Al-Dajani and Toksoz, 2001). In this case, with some loss of accuracy, one can convert the five fourth-order local effective parameters into three, thus approximating the inverted monoclinic layer as orthorhombic. Methods for computing the "nearest" orthorhombic approximations to an arbitrary anisotropic stiffness tensor are suggested by Kochetov and Slawinski (2009), Diner et al. (2011), and den Boer (2014).

CONCLUSION

In this work, we considered all types of waves propagating in a multilayer, elastic orthorhombic media, in which each layer shares the same horizontal symmetry plane but it is characterized by different azimuths of vertical symmetry planes. Based on the slowness-azimuth domain, fourth-order NMO velocity functions obtained in part 1 of this paper, in part 2, we derive the offset-azimuth domain fourth-order NMO velocity functions. The importance and applicability of the fourth-order NMO velocities in the different azimuthal domains are emphasized in part 1 of this paper.

The main result of this paper is the derived exact azimuthally dependent fourth-order NMO velocity functions (and hence the quartic coefficient and effective anellipticity functions) for all types of pure-mode and converted waves, without making any assumptions about weak anisotropy or acoustic approximations for P-waves. Once the exact solution is obtained, we also provided a simplified formula for a special case of weak azimuthal anisotropy, in which only the azimuthally anisotropic effective parameters are considered small and the relationship can therefore be linearized. In this case, the second- and fourth-order NMO velocities in all azimuthal domains converge to their simplest form in the slowness-azimuth/slowness domain. To further validate our result, we showed that the well-known formulas for a single layer, characterized by weak anisotropy and strong anisotropy, are particular cases of our exact multilayer solution.

Eight local (layer) and eight global (multilayer) effective parameters are proposed to define the azimuthally dependent fourth-order NMO velocity functions. The first three parameters refer to the second-order kinematic coefficients and the last five parameters refer to the fourth-order ones. The eight global effective parameters can also be classified into two different types of parameters: two azimuthally isotropic parameters and six azimuthally anisotropic parameters. We suggested an alternative notation, in which the global anisotropic effective parameters are normalized. The proposed eight effective parameters in both notations are generic model parameters and describe the kinematics of near-vertical waves in the slowness- and offset-azimuth domains. Note that the second- and fourth-order NMO velocity functions that use these effective parameters are different in the two domains.

To test the feasibility of our method, we introduced our derived azimuthally dependent fourth-order effective anellipticity function into the well-known offset-azimuth domain nonhyperbolic asymptotic traveltime approximation, and we compared the traveltimes with those obtained by a numerical ray tracing. The comparison shows an accurate match.

A major strength of the proposed method is that it can be extended to general azimuthal anisotropy. In part 1, we derived the eight local effective parameters of an elastic orthorhombic layer. All further derivations obtained in parts 1 and 2, such as the eight global effective parameters, the second-order azimuthal lags, and the phase- and offset-domain second- and fourth-order NMO velocity functions, hold for any type of anisotropy, provided that the reciprocity condition is in effect; i.e., the traveltime is an even function of the offset. This includes pure-mode waves in any anisotropic horizontally layered media (e.g., TTI, tilted orthorhombic, tilted monoclinic, and general triclinic), and in addition, converted waves for monoclinic media with horizontal plane of symmetry (and its particular case, orthorhombic).

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APPENDIX A

AZUMUTHAL LAG IN THE SLOWNESS-AZIMUTH DOMAIN AND OFFSET AZIMUTH VERSUS SLOWNESS AZIMUTH

The lag between the offset azimuth and slowness azimuth is given in equation 6, which includes the radial and transverse offset components. Applying equation 4 to these components, we obtain

$$\cos(\psi_{\mathrm{off}} - \psi_{\mathrm{slw}}) = \cos \Delta\psi_{0,\mathrm{slw}} - B_{\mathrm{slw}} \sin \Delta\psi_{0,\mathrm{slw}} p_h^2 + O(p_h^4),$$
$$\sin(\psi_{\mathrm{off}} - \psi_{\mathrm{slw}}) = \sin \Delta\psi_{0,\mathrm{slw}} + B_{\mathrm{slw}} \cos \Delta\psi_{0,\mathrm{slw}} p_h^2 + O(p_h^4), \tag{A-1}$$

where $\Delta\psi_{0,\mathrm{slw}} = \psi_{0,\mathrm{off}} - \psi_{0,\mathrm{slw}}$ is the zero-offset lag defined in the slowness-azimuth domain:

$$\cos \Delta\psi_{0,\mathrm{slw}} = \frac{U_2 + W_2 \cos 2(\psi_{\mathrm{slw}} - \Psi_{2,\mathrm{H}})}{\sqrt{U_2^2 + W_2^2 + 2U_2W_2 \cos 2(\psi_{\mathrm{slw}} - \Psi_{2,\mathrm{H}})}},$$
$$\sin \Delta\psi_{0,\mathrm{slw}} = \frac{-W_2 \sin 2(\psi_{\mathrm{slw}} - \Psi_{2,\mathrm{H}})}{\sqrt{U_2^2 + W_2^2 + 2U_2W_2 \cos 2(\psi_{\mathrm{slw}} - \Psi_{2,\mathrm{H}})}}, \tag{A-2}$$

where $\Psi_{2,\mathrm{H}}$ is the azimuth of the high second-order NMO velocity. We present parameter B_{slw} as the product of a kernel $K^{\mathrm{B}}_{\mathrm{slw}}$ and a factor

$$B_{\mathrm{slw}}(\psi_{\mathrm{slw}}) = \frac{1}{4}\frac{K^{\mathrm{B}}_{\mathrm{slw}}(\psi_{\mathrm{slw}})}{U_2^2 + W_2^2 + 2U_2W_2 \cos 2(\psi_{\mathrm{slw}} - \Psi_{2,\mathrm{H}})}. \tag{A-3}$$

The kernel can be presented in bilinear form

$$K^{\mathrm{B}}_{\mathrm{slw}}(\psi_{\mathrm{slw}}) = \mathbf{m}\mathbf{M}^{\mathrm{lag}}_{\mathrm{slw}}\mathbf{a}_{\mathrm{slw},5}, \tag{A-4}$$

where $\mathbf{m}$ is a row vector of length five that includes the high-order effective parameters

$$\mathbf{m} = [\, U_4 \quad W_{42x} \quad W_{42y} \quad W_{44x} \quad W_{44y} \,]. \tag{A-5}$$

The $\mathbf{a}_{\mathrm{slw},5}$ is a column vector of length five that depends on the slowness azimuth alone,

$$\mathbf{a}_{\mathrm{slw},5} = [1, \cos 2\psi_{\mathrm{slw}}, \sin 2\psi_{\mathrm{slw}}, \cos 4\psi_{\mathrm{slw}}, \sin 4\psi_{\mathrm{slw}}]^{\mathrm{T}}, \tag{A-6}$$

and $\mathbf{M}^{\mathrm{lag}}_{\mathrm{slw}}$ is a 5×5 matrix, whose components depend only on the low-order effective parameters,

$$\mathbf{M}^{\mathrm{lag}}_{\mathrm{slw}} = \begin{bmatrix} 0 & -4W_{2y} & +4W_{2x} & 0 & 0 \\ -3W_{2y} & 0 & -2U_2 & -W_{2y} & +W_{2x} \\ +3W_{2x} & +2U_2 & 0 & -W_{2x} & -W_{2y} \\ 0 & -4W_{2y} & -4W_{2x} & 0 & -4U_2 \\ 0 & +4W_{2x} & -4W_{2y} & +4U_2 & 0 \end{bmatrix}. \tag{A-7}$$

Finally, we express the offset azimuth (rather than the lag) in terms of the slowness azimuth, as shown in equation 7.

APPENDIX B

AZUMUTHAL LAG IN OFFSET-AZIMUTH DOMAIN AND SLOWNESS AZIMUTH VERSUS OFFSET AZIMUTH

In Appendix A, we obtained the offset azimuth and lag between the two azimuths as a function of the slowness azimuth. In this appendix, we derive the inverse relationship. The slowness azimuth and lag will be obtained in terms of the offset azimuth. Ignoring the high-order terms in equation A-1, we arrange it as

$$\cos(\psi_{\text{off}} - \psi_{\text{slw}}) = \cos \Delta\psi_{0,\text{slw}} + O(p_h^2),$$
$$\sin(\psi_{\text{off}} - \psi_{\text{slw}}) = \sin \Delta\psi_{0,\text{slw}} + O(p_h^2), \quad \text{(B-1)}$$

where the zero-offset terms of the azimuthal lag $\cos \Delta\psi_{0,\text{slw}}$ and $\sin \Delta\psi_{0,\text{slw}}$ are given in equation A-2. Equations A-2 and B-1 can be equivalently arranged as

$$\cos(\psi_{\text{off}}-\psi_{\text{slw}}) = \frac{U_2+W_{2x}\cos 2\psi_{\text{slw}}+W_{2y}\sin 2\psi_{\text{slw}}}{\sqrt{U_2^2+W_{2x}^2+W_{2y}^2+2U_2W_{2x}\cos 2\psi_{\text{slw}}+2U_2W_{2y}\sin 2\psi_{\text{slw}}}}+O(p_h^2),$$
$$\sin(\psi_{\text{off}}-\psi_{\text{slw}}) = \frac{W_{2y}\cos 2\psi_{\text{slw}}-W_{2x}\sin 2\psi_{\text{slw}}}{\sqrt{U_2^2+W_{2x}^2+W_{2y}^2+2U_2W_{2x}\cos 2\psi_{\text{slw}}+2U_2W_{2y}\sin 2\psi_{\text{slw}}}}+O(p_h^2). \quad \text{(B-2)}$$

We divide the second equation of this set by its first equation, and we solve the resulting relationship for the slowness azimuth. The result is

$$\cos \psi_{\text{slw}} = \cos \psi_{0,\text{slw}} + O(p_h^2),$$
$$\sin \psi_{\text{slw}} = \sin \psi_{0,\text{slw}} + O(p_h^2), \quad \text{(B-3)}$$

where the zero-offset terms of the slowness azimuth versus offset azimuth are presented in equation 10. The goal is to find the second-order terms, so we rewrite equation B-3 as

$$\cos \psi_{\text{slw}} = \cos \psi_{0,\text{slw}} + A_C p_h^2 + O(p_h^4),$$
$$\sin \psi_{\text{slw}} = \sin \psi_{0,\text{slw}} + A_S p_h^2 + O(p_h^4), \quad \text{(B-4)}$$

where A_C and A_S are the second-order coefficients for the sine and cosine of the slowness azimuth, respectively. Taking into account the fact that sine and cosine squared are linearly dependent, we conclude that A_C and A_S are not independent, and there is actually a single unknown coefficient. According to equation B-4,

$$\cos^2 \psi_{\text{slw}} + \sin^2 \psi_{\text{slw}} = \cos^2 \psi_{0,\text{slw}} + \sin^2 \psi_{0,\text{slw}} + 2(A_C \cos \psi_{\text{o,slw}} + A_S \sin \psi_{\text{o,slw}}) p_h^2 + O(p_h^4). \quad \text{(B-5)}$$

Because only the fourth-order error is allowed, we conclude that the following constraint holds to suppress the second-order error:

$$A_C \cos \psi_{0,\text{slw}} + A_S \sin \psi_{0,\text{slw}} = 0. \quad \text{(B-6)}$$

Therefore, without any loss of generality, we may assume

$$A_C = -B_{\text{off}} \sin \psi_{0,\text{slw}}, \quad A_S = +B_{\text{off}} \cos \psi_{0,\text{slw}}. \quad \text{(B-7)}$$

In this case, constraint B-6 is satisfied, and equation B-4 leads to equation 9, where B_{off} is an unknown expansion coefficient to be found. For this, we consider the ratio of the radial and transverse offset components h_R and h_T, which is the tangent of the azimuthal lag

$$\frac{h_T}{h_R} = \tan(\psi_{\text{off}} - \psi_{\text{slw}}) = \frac{\sin \psi_{\text{off}} \cos \psi_{\text{slw}} - \cos \psi_{\text{off}} \sin \psi_{\text{slw}}}{\cos \psi_{\text{off}} \cos \psi_{\text{slw}} + \sin \psi_{\text{off}} \sin \psi_{\text{slw}}}. \quad \text{(B-8)}$$

The offset components are presented in equation 4. We introduce this equation into equation B-8. Note that after this substitution, the lateral slowness p_h is canceled on the left side of equation B-8. The first- and third-order terms with p_h and p_h^3 in the numerator and denominator on the left side of equation B-8 become zero-order (slowness-independent) and second-order terms with p_h^2. Next, we expand the trigonometric functions of multiple arguments (such as $2\psi_{\text{slw}}$ and $4\psi_{\text{slw}}$) into the products and powers of trigonometric functions of a single argument ψ_{slw}. Then, we introduce the series expansion of equation 9 for the slowness azimuth into the equation obtained. Finally, we expand the resulting equation into the power series, up to the slowness squared (with the leading error terms of power four). This makes it possible to obtain the unknown expansion coefficient B_{off} in equation 9. The result may be presented as a product of a second-order factor and a fourth-order kernel

$$B_{\text{off}}(\psi_{\text{off}}) = \frac{1}{4}\frac{K_{\text{off}}^{\text{B}}(\psi_{\text{off}})}{[U_2^2+W_2^2-2U_2W_2\cos 2(\psi_{\text{off}}-\Psi_{2,\text{H}})]^2},$$
$$= \frac{1}{4}\frac{K_{\text{off}}^{\text{B}}(\psi_{\text{off}})}{(U_2^2+W_{2x}^2+W_{2y}^2-2U_2W_{2x}\cos 2\psi_{\text{off}}-2U_2W_{2y}\sin 2\psi_{\text{off}})^2}. \quad \text{(B-9)}$$

Like the slowness-domain case, the kernel can be presented in bilinear form

$$K_{\text{off}}^{\text{B}}(\psi_{\text{off}}) = \mathbf{m}\mathbf{M}_{\text{off}}^{\text{B}}\mathbf{a}_{\text{off},5}, \quad \text{(B-10)}$$

where $\mathbf{m}$ is a row vector of length five that includes the high-order effective parameters (equation A-5) and $\mathbf{a}_{\text{off},5}$ is a column vector of length five that depends on the offset azimuth alone

$$\mathbf{a}_{\text{off},5} = [1, \cos 2\psi_{\text{off}}, \sin 2\psi_{\text{off}}, \cos 4\psi_{\text{off}}, \sin 4\psi_{\text{off}}]^{\text{T}}. \quad \text{(B-11)}$$

The components of the matrix $\mathbf{M}_{\text{off}}^{\text{B}}$, 5×5, are listed below. Column 1 is

$$\begin{aligned} M_{\text{off},11}^{\text{B}} &= 0, \\ M_{\text{off},21}^{\text{B}} &= -3W_{2y}(U_2^2+W_{2x}^2+W_{2y}^2), \\ M_{\text{off},31}^{\text{B}} &= +3W_{2x}(U_2^2+W_{2x}^2+W_{2y}^2), \\ M_{\text{off},41}^{\text{B}} &= +24U_2W_{2x}W_{2y}, \\ M_{\text{off},51}^{\text{B}} &= -12U_2(W_{2x}^2-W_{2y}^2). \end{aligned} \quad \text{(B-12)}$$

Columns 2 and 3 are

$$\begin{aligned} &M_{\text{off},12}^{\text{B}} = +4W_{2y}(U_2^2+W_{2x}^2+W_{2y}^2), && M_{\text{off},13}^{\text{B}} = -4W_{2x}(U_2^2+W_{2x}^2+W_{2y}^2), \\ &M_{\text{off},22}^{\text{B}} = 0, && M_{\text{off},23}^{\text{B}} = +2U_2(U_2^2+3W_{2x}^2+3W_{2y}^2), \\ &M_{\text{off},32}^{\text{B}} = -2U_2(U_2^2+3W_{2x}^2+3W_{2y}^2), && M_{\text{off},33}^{\text{B}} = 0, \\ &M_{\text{off},42}^{\text{B}} = -4W_{2y}(3U_2^2+3W_{2x}^2-W_{2y}^2), && M_{\text{off},43}^{\text{B}} = -4W_{2x}(3U_2^2-W_{2x}^2+3W_{2y}^2), \\ &M_{\text{off},52}^{\text{B}} = +4W_{2x}(3U_2^2+W_{2x}^2-3W_{2y}^2), && M_{\text{off},53}^{\text{B}} = -4W_{2y}(3U_2^2-3W_{2x}^2+W_{2y}^2). \end{aligned} \quad \text{(B-13)}$$

Columns 4 and 5 are

$$\begin{aligned} M^{\rm B}_{{\rm off},14} &= -8U_2W_{2x}W_{2y}, & M^{\rm B}_{{\rm off},15} &= +4U_2(W^2_{2x}-W^2_{2y}),\\ M^{\rm B}_{{\rm off},24} &= +W_{2y}(3U^2_2+3W^2_{2x}-W^2_{2y}), & M^{\rm B}_{{\rm off},25} &= -W_{2x}(3U^2_2+W^2_{2x}-3W^2_{2y}),\\ M^{\rm B}_{{\rm off},34} &= +W_{2x}(3U^2_2-W^2_{2x}+3W^2_{2y}), & M^{\rm B}_{{\rm off},35} &= +W_{2y}(3U^2_2-3W^2_{2x}+W^2_{2y}),\\ M^{\rm B}_{{\rm off},44} &= 0, & M^{\rm B}_{{\rm off},45} &= +4U^3_2,\\ M^{\rm B}_{{\rm off},54} &= -4U^3_2, & M^{\rm B}_{{\rm off},55} &= 0. \end{aligned} \tag{B-14}$$

After we found the second-order approximation for the slowness azimuth versus offset azimuth, we can now also approximate the lag between the two azimuths as a function of the offset azimuth

$$\begin{aligned} \cos\Delta\psi &= \cos\Delta\psi_{0,{\rm off}} + B_{\rm off}\,\sin\Delta\psi_{0,{\rm off}}\,p^2_h + O(p^4_h),\\ \sin\Delta\psi &= \sin\Delta\psi_{0,{\rm off}} - B_{\rm off}\,\cos\Delta\psi_{0,{\rm off}}\,p^2_h + O(p^4_h), \end{aligned} \tag{B-15}$$

where the zero-offset lag versus the offset azimuth reads

$$\begin{aligned} \cos\Delta\psi_{0,{\rm off}} &= \frac{U_2 - W_{2x}\cos 2\psi_{\rm off} - W_{2y}\sin 2\psi_{\rm off}}{\sqrt{U^2_2+W^2_{2x}+W^2_{2y}-2U_2W_{2x}\cos 2\psi_{\rm off}-2U_2W_{2y}\sin 2\psi_{\rm off}}},\\ \sin\Delta\psi_{0,{\rm off}} &= \frac{W_{2y}\cos 2\psi_{\rm off} - W_{2x}\sin 2\psi_{\rm off}}{\sqrt{U^2_2+W^2_{2x}+W^2_{2y}-2U_2W_{2x}\cos 2\psi_{\rm off}-2U_2W_{2y}\sin 2\psi_{\rm off}}}, \end{aligned} \tag{B-16}$$

or equivalently the zero-offset lag may be arranged as

$$\begin{aligned} \cos\Delta\psi_{0,{\rm off}} &= \frac{U_2 - W_2\cos 2(\psi_{\rm off}-\Psi_{2,{\rm H}})}{\sqrt{U^2_2+W^2_2-2U_2W_2\cos 2(\psi_{\rm off}-\Psi_{2,{\rm H}})}},\\ \sin\Delta\psi_{0,{\rm off}} &= \frac{-W_2\sin 2(\psi_{\rm off}-\Psi_{2,{\rm H}})}{\sqrt{U^2_2+W^2_2-2U_2W_2\cos 2(\psi_{\rm off}-\Psi_{2,{\rm H}})}}, \end{aligned} \tag{B-17}$$

and parameter $B_{\rm off}$ is defined in equation B-9.

APPENDIX C

FOURTH-ORDER NMO VELOCITY IN THE OFFSET-AZIMUTH/SLOWNESS DOMAIN

It follows from equations 4, 9, and 11 that the second-order NMO velocity in the offset-azimuth/slowness domain reads

$$\begin{aligned} V^2_{2,{\rm slw}}(\psi_{\rm off})t_{\rm o} &= \frac{(U^2_2-W^2_{2x}-W^2_{2y})(U_2-W_{2x}\cos 2\psi_{\rm off}-W_{2y}\sin 2\psi_{\rm off})}{U^2_2+W^2_{2x}+W^2_{2y}-2U_2W_{2x}\cos 2\psi_{\rm off}-2U_2W_{2y}\sin 2\psi_{\rm off}},\\ &= \frac{(U^2_2-W^2_2)[U_2-W_2\cos 2(\psi_{\rm off}-\Psi_{2,{\rm H}})]}{U^2_2+W^2_2-2U_2W_2\cos 2(\psi_{\rm off}-\Psi_{2,{\rm H}})}, \end{aligned} \tag{C-1}$$

or alternatively

$$V^2_{2,{\rm slw}}(\psi_{\rm off}) = V^2_{2,{\rm H}}V^2_{2,{\rm L}}\frac{V^2_{2,{\rm H}}\sin^2(\psi_{\rm off}-\Psi_{2,{\rm H}})+V^2_{2,{\rm L}}\cos^2(\psi_{\rm off}-\Psi_{2,{\rm H}})}{V^4_{2,{\rm H}}\sin^2(\psi_{\rm off}-\Psi_{2,{\rm H}})+V^4_{2,{\rm L}}\cos^2(\psi_{\rm off}-\Psi_{2,{\rm H}})}. \tag{C-2}$$

The fourth-order NMO velocity is presented in a factorized form (equation 12). The second-order scaling factor is given by

$$\begin{aligned} K_{2,{\rm slw}}(\psi_{\rm off}) &= \frac{t^{-1}_{\rm o}}{3(U^2_2+W^2_{2x}+W^2_{2y}-2U_2W_{2x}\cos 2\psi_{\rm off}-2U_2W_{2y}\sin 2\psi_{\rm off})^3},\\ &= \frac{t^{-1}_{\rm o}}{3[U^2_2+W^2_2-2U_2W_2\cos 2(\psi_{\rm off}-\Psi_{2,{\rm H}})]^3}, \end{aligned} \tag{C-3}$$

or, alternatively,

$$K_{2,{\rm slw}}(\psi_{\rm off}) = \frac{1}{3}\frac{t^{-7}_{\rm o}}{[V^4_{2,{\rm H}}\sin^2(\psi_{\rm off}-\Psi_{2,{\rm H}})+V^4_{2,{\rm L}}\cos^2(\psi_{\rm off}-\Psi_{2,{\rm H}})]^3}. \tag{C-4}$$

For any offset azimuth, the fourth-order kernel is a scalar value presented in bilinear form (equation 13), where $\mathbf{m}$ is a row vector that includes the five fourth-order global effective parameters (equation A-5), and $\mathbf{a}_{{\rm off},7}$ is a column vector of length seven that depends on the offset azimuth alone,

$$\begin{aligned} \mathbf{a}_{{\rm off},7} = [&1, \cos 2\psi_{\rm off}, \sin 2\psi_{\rm off}, \cos 4\psi_{\rm off}, \sin 4\psi_{\rm off},\\ &\cos 6\psi_{\rm off}, \sin 6\psi_{\rm off}]^{\rm T}. \end{aligned} \tag{C-5}$$

The $\mathbf{M}_{\rm off/slw}$ is 5×7 matrix needed to compute the kernel $K_{4,{\rm slw}}(\psi_{\rm off})$. Its components depend on the second-order global effective parameters. The first column reads

$$\begin{aligned} M_{{\rm off/slw},11} &= 2(U^2_2+W^2_{2x}+W^2_{2y})\ [3U^4_2+22U^2_2(W^2_{2x}+W^2_{2y})+5(W^2_{2x}+W^2_{2y})^2],\\ M_{{\rm off/slw},21} &= -2U_2W_{2x}\ [11U^4_2+34U^2_2(W^2_{2x}+W^2_{2y})+15(W^2_{2x}+W^2_{2y})^2],\\ M_{{\rm off/slw},31} &= -2U_2W_{2y}\ [11U^4_2+34U^2_2(W^2_{2x}+W^2_{2y})+15(W^2_{2x}+W^2_{2y})^2],\\ M_{{\rm off/slw},41} &= 4(W^2_{2x}-W^2_{2y})\ [12U^4_2+19U^2_2(W^2_{2x}+W^2_{2y})-(W^2_{2x}+W^2_{2y})^2],\\ M_{{\rm off/slw},51} &= 8W_{2x}W_{2y}\ [12U^4_2+19U^2_2(W^2_{2x}+W^2_{2y})-(W^2_{2x}+W^2_{2y})^2]. \end{aligned} \tag{C-6}$$

The second column reads

$$\begin{aligned} M_{{\rm off/slw},12} &= -4U_2W_{2x}[9U^4_2+26U^2_2(W^2_{2x}+W^2_{2y})+10(W^2_{2x}+W^2_{2y})^2],\\ M_{{\rm off/slw},22} &= +10(W^2_{2x}-W^2_{2y})(W^2_{2x}+W^2_{2y})^2-3(W^2_{2x}+W^2_{2y})^3\\ &\quad +2U^2_2(43W^2_{2x}-4W^2_{2y})(W^2_{2x}+W^2_{2y})+3U^4_2(27W^2_{2x}+5W^2_{2y})+6U^6_2,\\ M_{{\rm off/slw},32} &= 2W_{2x}W_{2y}[33U^4_2+47U^2_2(W^2_{2x}+W^2_{2y})+10(W^2_{2x}+W^2_{2y})^2],\\ M_{{\rm off/slw},42} &= -2U_2W_{2x}[13U^4_2+2U^2_2(31W^2_{2x}-17W^2_{2y})+3(5W^2_{2x}-23W^2_{2y})(W^2_{2x}+W^2_{2y})],\\ M_{{\rm off/slw},52} &= -2U_2W_{2y}[13U^4_2+2U^2_2(55W^2_{2x}+7W^2_{2y})+3(19W^2_{2x}-9W^2_{2y})(W^2_{2x}+W^2_{2y})]. \end{aligned} \tag{C-7}$$

The third column reads

$$\begin{aligned} M_{{\rm off/slw},13} &= -4U_2W_{2y}[9U^4_2+26U^2_2(W^2_{2x}+W^2_{2y})+10(W^2_{2x}+W^2_{2y})^2],\\ M_{{\rm off/slw},23} &= 2W_{2x}W_{2y}[33U^4_2+47U^2_2(W^2_{2x}+W^2_{2y})+10(W^2_{2x}+W^2_{2y})^2],\\ M_{{\rm off/slw},33} &= -10(W^2_{2x}-W^2_{2y})(W^2_{2x}+W^2_{2y})^2-3(W^2_{2x}+W^2_{2y})^3\\ &\quad -2U^2_2(4W^2_{2x}-43W^2_{2y})(W^2_{2x}+W^2_{2y})+3U^4_2(5W^2_{2x}+27W^2_{2y})+6U^6_2,\\ M_{{\rm off/slw},43} &= +2U_2W_{2y}[13U^4_2-2U^2_2(17W^2_{2x}-31W^2_{2y})-3(23W^2_{2x}-5W^2_{2y})(W^2_{2x}+W^2_{2y})],\\ M_{{\rm off/slw},53} &= -2U_2W_{2x}[13U^4_2+2U^2_2(7W^2_{2x}+55W^2_{2y})-3(9W^2_{2x}-19W^2_{2y})(W^2_{2x}+W^2_{2y})]. \end{aligned} \tag{C-8}$$

The fourth column reads

$$\begin{aligned}
M_{\mathrm{off/slw},14} &= 4(W_{2x}^2 - W_{2y}^2)(U_2^2 + W_{2x}^2 + W_{2y}^2)(10U_2^2 - W_{2x}^2 - W_{2y}^2),\\
M_{\mathrm{off/slw},24} &= -2U_2W_{2x}[7U_2^4 + 2U_2^2(13W_{2x}^2 - 11W_{2y}^2) + 3(W_{2x}^2 - 7W_{2y}^2)(W_{2x}^2 + W_{2y}^2)],\\
M_{\mathrm{off/slw},34} &= +2U_2W_{2y}[7U_2^4 - 2U_2^2(11W_{2x}^2 - 13W_{2y}^2) - 3(7W_{2x}^2 - W_{2y}^2)(W_{2x}^2 + W_{2y}^2)],\\
M_{\mathrm{off/slw},44} &= 10(W_{2x}^2 + W_{2y}^2)(W_{2x}^4 - 6W_{2x}^2W_{2y}^2 + W_{2y}^4)\\
&\quad + 2U_2^2(7W_{2x}^4 - 90W_{2x}^2W_{2y}^2 + 7W_{2y}^4) + 42U_2^4(W_{2x}^2 + W_{2y}^2) + 6U_2^6,\\
M_{\mathrm{off/slw},54} &= +8W_{2x}W_{2y}(W_{2x}^2 - W_{2y}^2)[13U_2^2 + 5(W_{2x}^2 + W_{2y}^2)].
\end{aligned} \tag{C-9}$$

The fifth column reads

$$\begin{aligned}
M_{\mathrm{off/slw},15} &= 8W_{2x}W_{2y}(U_2^2 + W_{2x}^2 + W_{2y}^2)(10U_2^2 - W_{2x}^2 - W_{2y}^2)\\
M_{\mathrm{off/slw},25} &= -2U_2W_{2y}[7U_2^4 + 2U_2^2(25W_{2x}^2 + W_{2y}^2) + 3(5W_{2x}^2 - 3W_{2y}^2)(W_{2x}^2 + W_{2y}^2)]\\
M_{\mathrm{off/slw},35} &= -2U_2W_{2x}[7U_2^4 + 2U_2^2(W_{2x}^2 + 25W_{2y}^2) - 3(3W_{2x}^2 - 5W_{2y}^2)(W_{2x}^2 + W_{2y}^2)],\\
M_{\mathrm{off/slw},45} &= 8W_{2x}W_{2y}(W_{2x}^2 - W_{2y}^2)[13U_2^2 + 5(W_{2x}^2 + W_{2y}^2)],\\
M_{\mathrm{off/slw},55} &= -10(W_{2x}^2 + W_{2y}^2)(W_{2x}^4 - 6W_{2x}^2W_{2y}^2 + W_{2y}^4) - 24U_2^2(W_{2x}^2 + W_{2y}^2)^2\\
&\quad - 2U_2^2(7W_{2x}^4 - 90W_{2x}^2W_{2y}^2 + 7W_{2y}^4) + 42U_2^4(W_{2x}^2 + W_{2y}^2) + 6U_2^6.
\end{aligned} \tag{C-10}$$

The sixth column reads

$$\begin{aligned}
M_{\mathrm{off/slw},16} &= -4U_2W_{2x}(4U_2^2 - W_{2x}^2 - W_{2y}^2)(W_{2x}^2 - 3W_{2y}^2),\\
M_{\mathrm{off/slw},26} &= -(W_{2x}^2 + W_{2y}^2)(W_{2x}^4 - 6W_{2x}^2W_{2y}^2 + W_{2y}^4)\\
&\quad + 2U_2^2(2W_{2x}^4 - 21W_{2x}^2W_{2y}^2 + 5W_{2y}^4) + 9U_2^4(W_{2x}^2 - W_{2y}^2),\\
M_{\mathrm{off/slw},36} &= -2W_{2x}W_{2y}[9U_2^4 - U_2^2(17W_{2x}^2 - 11W_{2y}^2) + 2(W_{2x}^2 - W_{2y}^2)(W_{2x}^2 + W_{2y}^2)],\\
M_{\mathrm{off/slw},46} &= -2U_2W_{2x}[5U_2^4 - 2U_2^2(W_{2x}^2 + W_{2y}^2) + 3(W_{2x}^4 - 10W_{2x}^2W_{2y}^2 + 5W_{2y}^4)],\\
M_{\mathrm{off/slw},56} &= +2U_2W_{2y}[5U_2^4 - 2U_2^2(W_{2x}^2 + W_{2y}^2) - 3(5W_{2x}^4 - 10W_{2x}^2W_{2y}^2 + W_{2y}^4)].
\end{aligned} \tag{C-11}$$

The seventh column reads

$$\begin{aligned}
M_{\mathrm{off/slw},17} &= -4U_2W_{2y}(4U_2^2 - W_{2x}^2 - W_{2y}^2)(3W_{2x}^2 - W_{2y}^2),\\
M_{\mathrm{off/slw},27} &= +2W_{2x}W_{2y}[9U_2^4 + U_2^2(11W_{2x}^2 - 17W_{2y}^2) - 2(W_{2x}^2 - W_{2y}^2)(W_{2x}^2 + W_{2y}^2)],\\
M_{\mathrm{off/slw},37} &= +(W_{2x}^2 + W_{2y}^2)(W_{2x}^4 - 6W_{2x}^2W_{2y}^2 + W_{2y}^4)\\
&\quad - 2U_2^2(5W_{2x}^4 - 21W_{2x}^2W_{2y}^2 + 2W_{2y}^4) + 9U_2^4(W_{2x}^2 - W_{2y}^2),\\
M_{\mathrm{off/slw},47} &= -2U_2W_{2y}[5U_2^4 - 2U_2^2(W_{2x}^2 + W_{2y}^2) + 3(5W_{2x}^4 - 10W_{2x}^2W_{2y}^2 + W_{2y}^4)],\\
M_{\mathrm{off/slw},57} &= -2U_2W_{2x}[5U_2^4 - 2U_2^2(W_{2x}^2 + W_{2y}^2) - 3(W_{2x}^4 - 10W_{2x}^2W_{2y}^2 + 5W_{2y}^4)].
\end{aligned} \tag{C-12}$$

APPENDIX D

FOURTH-ORDER NMO VELOCITY IN THE OFFSET-AZIMUTH/OFFSET DOMAIN

To derive the fourth-order NMO velocity function in the offset-azimuth/offset domain, we use equation 4 for the radial and transverse offset components and the traveltime. In these relationships, we expand the trigonometric functions of multiple arguments (such as $2\psi_{\mathrm{slw}}$ and $4\psi_{\mathrm{slw}}$) into the products and powers of the functions of a single argument ψ_{slw}. Next, we introduce the series expansion for the slowness azimuth (equation 9) into each of the equations of equation set 4. At this time, parameter B_{off} is already a known value. The NMO velocity can be found by applying the limit in the first equation of equation set 2, leading to

$$V_2^2 t_{\mathrm{o}} = \frac{U_2^2 - W_{2x}^2 - W_{2y}^2}{U_2 - W_{2x}\cos 2\psi_{\mathrm{off}} - W_{2y}\sin 2\psi_{\mathrm{off}}} = \frac{U_2^2 - W_2^2}{U_2 - W_2\cos 2(\psi_{\mathrm{off}} - \Psi_{2,\mathrm{H}})}. \tag{D-1}$$

Taking into account equation 19 of Part 1, equation C-1 can be simplified to the elliptic dependency obtained by Grechka and Tsvankin (1998, 1999):

$$\frac{1}{V_{2,\mathrm{off}}^2(\psi_{\mathrm{off}})} = \frac{\sin^2(\psi_{\mathrm{off}} - \Psi_{2,\mathrm{H}})}{V_{2,\mathrm{L}}^2} + \frac{\cos^2(\psi_{\mathrm{off}} - \Psi_{2,\mathrm{H}})}{V_{2,\mathrm{H}}^2}. \tag{D-2}$$

The quartic coefficient A_4 of the moveout can be obtained by applying the limit in the second equation of set 2. Equation 3 then relates the quartic coefficient A_4 to the fourth-order NMO velocity V_4. The result may be presented as a product of the normalizing factor and the fourth-order kernel, as shown in equation 14. The normalizing factor depends on the second-order parameters only

$$K_{2,\mathrm{off}}(\psi_{\mathrm{off}}) = \frac{2t_{\mathrm{o}}^{-1}}{(U_2 - W_{2x}\cos 2\psi_{\mathrm{off}} - W_{2y}\sin 2\psi_{\mathrm{off}})^4} = \frac{2t_{\mathrm{o}}^{-1}}{[U_2 - W_2\cos 2(\psi_{\mathrm{off}} - \Psi_{2,\mathrm{H}})]^4}, \tag{D-3}$$

and it may also be arranged as

$$K_{2,\mathrm{off}}(\psi_{\mathrm{off}}) = \frac{2t_{\mathrm{o}}^{-5}}{[V_{2,\mathrm{L}}^2\cos^2(\psi_{\mathrm{off}} - \Psi_{2,\mathrm{H}}) + V_{2,\mathrm{H}}^2\sin(\psi_{\mathrm{off}} - \Psi_{2,\mathrm{H}})]^4}. \tag{D-4}$$

Like the slowness-azimuth/offset fourth-order kernel $K_{4,\mathrm{off}}(\psi_{\mathrm{slw}})$, the offset-azimuth/offset fourth-order kernel $K_{4,\mathrm{off}}(\psi_{\mathrm{off}})$ can also be presented in bilinear form, as shown in equation 15. Parameter $\mathbf{m}$ is a row vector of length five that includes the fourth-order effective parameters (equation A-5), $\mathbf{a}_{\mathrm{off},5}$ is a column vector of length five that depends on the offset azimuth alone (equation B-11), and $\mathbf{M}_{\mathrm{off/off}}$ is a matrix of dimension 5×5, whose components depend on the fourth-order parameters. Column 1 reads

$$\begin{aligned}
M_{\mathrm{off/off},11} &= +U_2^4 + 4U_2^2(W_{2x}^2 + W_{2y}^2) + (W_{2x}^2 + W_{2y}^2)^2,\\
M_{\mathrm{off/off},21} &= +3U_2W_{2x}(U_2^2 + W_{2x}^2 + W_{2y}^2),\\
M_{\mathrm{off/off},31} &= +3U_2W_{2y}(U_2^2 + W_{2x}^2 + W_{2y}^2),\\
M_{\mathrm{off/off},41} &= +6U_2^2(W_{2x}^2 - W_{2y}^2),\\
M_{\mathrm{off/off},51} &= +12U_2^2W_{2x}W_{2y}.
\end{aligned} \tag{D-5}$$

Columns 2 and 3 read

$$\begin{aligned}
M_{\mathrm{off/off},12} &= +4U_2W_{2x}(U_2^2 + W_{2x}^2 + W_{2y}^2), & M_{\mathrm{off/off},13} &= +4U_2W_{2y}(U_2^2 + W_{2x}^2 + W_{2y}^2),\\
M_{\mathrm{off/off},22} &= +U_2^4 + W_{2x}^4 + 6U_2^2W_{2x}^2 - W_{2y}^4, & M_{\mathrm{off/off},23} &= +2W_{2x}W_{2y}(3U_2^2 + W_{2x}^2 + W_{2y}^2),\\
M_{\mathrm{off/off},32} &= +2W_{2x}W_{2y}(3U_2^2 + W_{2x}^2 + W_{2y}^2), & M_{\mathrm{off/off},33} &= +U_2^4 - W_{2x}^4 + 6U_2^2W_{2y}^2 + W_{2y}^4,\\
M_{\mathrm{off/off},42} &= +4U_2W_{2x}(U_2^2 + W_{2x}^2 - 3W_{2y}^2), & M_{\mathrm{off/off},43} &= -4U_2W_{2y}(U_2^2 - 3W_{2x}^2 + W_{2y}^2),\\
M_{\mathrm{off/off},52} &= +4U_2W_{2y}(U_2^2 + 3W_{2x}^2 - W_{2y}^2), & M_{\mathrm{off/off},53} &= +4U_2W_{2x}(U_2^2 - W_{2x}^2 + 3W_{2y}^2).
\end{aligned} \tag{D-6}$$

Columns 4 and 5 read

$$
\begin{aligned}
&M_{\mathrm{off/off},14}=+2U_2^2(W_{2x}^2-W_{2y}^2), && M_{\mathrm{off/off},15}=+4U_2^2W_{2x}W_{2y},\\
&M_{\mathrm{off/off},24}=+U_2W_{2x}(U_2^2+W_{2x}^2-3W_{2y}^2), && M_{\mathrm{off/off},25}=+U_2W_{2y}(U_2^2+3W_{2x}^2-W_{2y}^2),\\
&M_{\mathrm{off/off},34}=-U_2W_{2y}(U_2^2-3W_{2x}^2+W_{2y}^2), && M_{\mathrm{off/off},35}=+U_2W_{2x}(U_2^2-W_{2x}^2+3W_{2y}^2),\\
&M_{\mathrm{off/off},44}=+U_2^4+W_{2x}^4-6W_{2x}^2W_{2y}^2+W_{2y}^4, && M_{\mathrm{off/off},45}=+4W_{2x}W_{2y}(W_{2x}^2-W_{2y}^2),\\
&M_{\mathrm{off/off},54}=+4W_{2x}W_{2y}(W_{2x}^2-W_{2y}^2), && M_{\mathrm{off/off},55}=+U_2^4-W_{2x}^4+6W_{2x}^2W_{2y}^2-W_{2y}^4.
\end{aligned}
\tag{D-7}
$$

Table D-1 lists the units of factors needed to compute the fourth-order velocity in the slowness-azimuth/offset, offset-azimuth/slowness, and offset-azimuth/offset domains.

Table D-1. Units of fourth-order velocity factors in the three azimuthal domains.

Entry	Slowness-azimuth/offset	Offset-azimuth/slowness	Offset-azimuth/offset
K_2	$\mathrm{s/m^4}$	$\mathrm{s^5/m^{12}}$	$\mathrm{s^3/m^8}$
K_4	$\mathrm{m^8/s^5}$	$\mathrm{m^{16}/s^9}$	$\mathrm{m^{12}/s^7}$
L	$\mathrm{m^4/s^3}$	$\mathrm{m^4/s^3}$	$\mathrm{m^4/s^3}$
M	$\mathrm{m^4/s^2}$	$\mathrm{m^{12}/s^6}$	$\mathrm{m^8/s^4}$
P	Unitless	Unitless	Unitless
A_4	Unitless	Unitless	Unitless
V_4^4	$\mathrm{m^4/s^4}$	$\mathrm{m^4/s^4}$	$\mathrm{m^4/s^4}$

APPENDIX E

NORMALIZED SET OF GLOBAL EFFECTIVE PARAMETERS

Our proposed eight global effective parameters can be classified into two parameter subsets: azimuthally isotropic $\{U_2, U_4\}$ and azimuthally anisotropic $\{W_{2x}, W_{2y}, W_{42x}, W_{42y}, W_{44x}, W_{44y}\}$. The main advantages in using these parameter subsets are that they are generic parameters of the physically layered model, defining the second- and fourth-order NMO velocity functions in the slowness-azimuth and offset-azimuth domains. Furthermore, they provide simple forward and inverse Dix-type transforms. In this appendix, we propose normalizing the global effective parameters to provide more intuitive and convenient interpretation values, which are preferable when trying to invert them from seismic data. The alternative eight parameters are numerated with closed brackets, (1), (2), …, (8).

Second-order effective parameters

For the second-order effective parameters, we propose the following three alternative effective parameters (see also equation 19 of part 1):

$$
\begin{aligned}
&(1)\ \overline{V}_2^2=\frac{U_2}{t_o},\\
&(2)\ e_2=\frac{W_2}{U_2},\quad W_2=\sqrt{W_{2x}^2+W_{2y}^2},\\
&(3)\ \Psi_{2,\mathrm{H}},\quad \cos 2\Psi_{2,\mathrm{L}}=\frac{W_{2x}}{W_2},\quad \sin 2\Psi_{2,\mathrm{H}}=\frac{W_{2y}}{W_2},
\end{aligned}
\tag{E-1}
$$

where $\overline{V}_2$ is the azimuthally isotropic second-order NMO velocity, e_2 is the effective elliptic parameter, and $\Psi_{2,\mathrm{H}}$ is the effective azimuth of the high second-order NMO velocity. Hence, the high and low NMO velocities characterizing the NMO velocities in the direction of $\Psi_{2,\mathrm{H}}$ and in its perpendicular direction $\Psi_{2,\mathrm{L}}=\Psi_{2,\mathrm{H}}+\pi/2$, respectively, are given by

$$
\begin{aligned}
V_{2,\mathrm{H}}^2&=\frac{U_2+W_2}{t_o}=\bar{V}_2^2(1+e_2),\\
V_{2,\mathrm{L}}^2&=\frac{U_2-W_2}{t_o}=\bar{V}_2^2(1-e_2).
\end{aligned}
\tag{E-2}
$$

Fourth-order effective parameters

For the fourth-order effective parameters, we propose the following five alternative effective parameters. The azimuthally isotropic fourth-order NMO velocity is defined by

$$
\overline{V}_4^4=\frac{2U_4}{t_o},
\tag{E-3}
$$

and this leads to the normalized azimuthally isotropic effective anellipticity

$$
(4)\ \bar{\eta}_{\mathrm{eff}}=\frac{\overline{V}_4^4-\overline{V}_2^4}{8\overline{V}_2^4}=\frac{2U_4t_o-U_2^2}{8U_2^2},\qquad U_4=\frac{1+8\bar{\eta}_{\mathrm{eff}}}{2t_o}U_2^2.
\tag{E-4}
$$

For the azimuths of the low and high NMO velocities, $\Psi_{2,\mathrm{H}}$ and $\Psi_{2,\mathrm{H}}+\pi/2$, the relationships for the fourth-order NMO velocities are simplified and become domain-independent:

$$
\begin{aligned}
V_{4,\mathrm{H}}^4&=\frac{2}{t_o}\left[U_4+\frac{W_{2x}W_{42x}+W_{2y}W_{42y}}{\sqrt{W_{2x}^2+W_{2y}^2}}\right.\\
&\qquad\left.+\frac{(W_{2x}^2-W_{2y}^2)W_{44x}+2W_{2x}W_{2y}W_{44y}}{W_{2x}^2+W_{2y}^2}\right],\\
V_{4,\mathrm{L}}^4&=\frac{2}{t_o}\left[U_4-\frac{W_{2x}W_{42x}+W_{2y}W_{42y}}{\sqrt{W_{2x}^2+W_{2y}^2}}\right.\\
&\qquad\left.+\frac{(W_{2x}^2-W_{2y}^2)W_{44x}+2W_{2x}W_{2y}W_{44y}}{W_{2x}^2+W_{2y}^2}\right].
\end{aligned}
\tag{E-5}
$$

The corresponding azimuthally anisotropic effective anellipticities are then given by

$$
\eta_{\mathrm{eff,L}}=\frac{V_{4,\mathrm{L}}^4-V_{2,\mathrm{L}}^4}{8V_{2,\mathrm{L}}^4},\qquad \eta_{\mathrm{eff,H}}=\frac{V_{4,\mathrm{H}}^4-V_{2,\mathrm{H}}^4}{8V_{2,\mathrm{H}}^4}.
\tag{E-6}
$$

Next, we introduce the "high" and "low" residual effective anellipticities

$$
\begin{aligned}
&(5)\ e_{4,\mathrm{L}}=\eta_{\mathrm{eff,L}}-\bar{\eta}_{\mathrm{eff}},\\
&(6)\ e_{4,\mathrm{H}}=\eta_{\mathrm{eff,H}}-\bar{\eta}_{\mathrm{eff}}.
\end{aligned}
\tag{E-7}
$$

The last two effective parameters are the additional azimuths related to 2ψ and 4ψ:

$$\cos 2\Psi_{42}=\frac{W_{42x}}{W_{42}},\ \sin 2\Psi_{42}=\frac{W_{42y}}{W_{42}},\ W_{42}=\pm\sqrt{W_{42x}^2+W_{42y}^2},$$
$$\cos 4\Psi_{44}=\frac{W_{44x}}{W_{44}},\ \sin 4\Psi_{44}=\frac{W_{44y}}{W_{44}},\ W_{44}=\pm\sqrt{W_{44x}^2+W_{44y}^2}. \tag{E-8}$$

The signs of W_{42} and W_{44} are chosen, so that the fourth-order azimuths Ψ_{42} and Ψ_{44} are the closest to the second-order azimuth $\Psi_{2,\mathrm{H}}$, and for a single-layer model, $\Psi_{42}=\Psi_{44}=\Psi_{2,\mathrm{H}}$. Thus, it is convenient to choose the last two effective azimuth parameters as

$$(7)\ \Delta\Psi_{42}=\Psi_{42}-\Psi_{2,\mathrm{H}},$$
$$(8)\ \Delta\Psi_{44}=\Psi_{44}-\Psi_{2,\mathrm{H}}. \tag{E-9}$$

To obtain the inverse relationships, we introduce equations E-1 and E-8 into equation E-5, and this leads to

$$U_4+W_{42}\cos 2\Delta\Psi_{42}+W_{44}\cos 4\Delta\Psi_{44}=\frac{V_{4,\mathrm{H}}^4 t_o}{2},$$
$$U_4-W_{42}\cos 2\Delta\Psi_{42}+W_{44}\cos 4\Delta\Psi_{44}=\frac{V_{4,\mathrm{L}}^4 t_o}{2}. \tag{E-10}$$

The solution reads

$$W_{42}=\frac{V_{4,\mathrm{H}}^4-V_{4,\mathrm{L}}^4}{4\cos 2\Delta\Psi_{42}}t_o,$$
$$W_{44}=\frac{(V_{4,\mathrm{L}}^4+V_{4,\mathrm{H}}^4)t_o-4U_4}{4\cos 4\Delta\Psi_{44}}=\frac{V_{4,\mathrm{L}}^4-2\bar{V}_4^4+V_{4,\mathrm{H}}^4}{4\cos 4\Delta\Psi_{44}}t_o, \tag{E-11}$$

and equation E-8 is then applied to find $W_{42x}, W_{42y}, W_{44x}$, and W_{44y}. Note that in the case of a single layer, there are no additional azimuths and the cosines in the denominators of equation E-11 are equal to one. For a single layer, equations E-10 and E-11 simplify to

$$u_4+w_{42}+w_{44}=\frac{v_{4,\mathrm{H}}^4\Delta t_o}{2},\quad u_4-w_{42}+w_{44}=\frac{v_{4,\mathrm{L}}^4\Delta t_o}{2}, \tag{E-12}$$

$$w_{42}=\frac{v_{4,\mathrm{H}}^4-v_{4,\mathrm{L}}^4}{4}\Delta t_o,\quad w_{44}=\frac{v_{4,\mathrm{L}}^4+v_{4,\mathrm{H}}^4}{4}\Delta t_o-u_4. \tag{E-13}$$

We emphasize that $V_{2,\mathrm{H}}$ and $V_{2,\mathrm{L}}$ are the highest and the lowest second-order NMO velocities, whereas this is not true for $V_{4,\mathrm{H}}$ and $V_{4,\mathrm{L}}$, which are the fourth-order NMO velocities in the directions of $\Psi_{2,\mathrm{H}}$ and $\Psi_{2,\mathrm{L}}$, respectively. Moreover, it is not necessarily true that $V_{4,\mathrm{H}}>V_{4,\mathrm{L}}$, and for a multilayer medium, $V_{4,\mathrm{H}}$ and $V_{4,\mathrm{L}}$ are not the extreme values of function $V_4(\psi)$ (see Figure 2). However, for a single layer, these are extreme values due to the existence of the vertical symmetry planes (see Figure 1).

Finally, we provide representative values of the effective parameters computed for the single-layer (homogeneous) orthorhombic model considered in the section of the numerical examples (see the medium properties in Table 1 of part 1). The original effective parameters are

$$\begin{aligned}
&t_o=0.285714\,\mathrm{s},\\
&U_2=4.72500\,\mathrm{km^2/s},\quad W_{2x}=-0.52500\,\mathrm{km^2/s}\quad W_{2y}=0,\\
&U_4=45.6371\,\mathrm{km^4/s^3},\quad W_{42x}=-10.8090\,\mathrm{km^4/s^3}\quad W_{42y}=0,\\
&\qquad W_{44x}=6.90357\,\mathrm{km^4/s^3}\quad W_{44y}=0,
\end{aligned} \tag{E-14}$$

and the normalized effective parameters are

$$\begin{aligned}
&\bar{v}_2=4.06630\ \mathrm{km/s}, && e_2=0.111111,\\
&v_{2,\mathrm{H}}=4.28661\ \mathrm{km/s}, && v_{2,\mathrm{L}}=3.83406\ \mathrm{km/s},\\
&v_{4,\mathrm{H}}=4.58892\ \mathrm{km/s}, && v_{4,\mathrm{L}}=4.13420\ \mathrm{km/s},\\
&e_{4,\mathrm{H}}=1.81602\times10^{-2}, && e_{4,\mathrm{L}}=2.29703\times10^{-2},\\
&\bar{\eta}_{\mathrm{eff}}=2.10112\times10^{-2}, && \Psi_{2,\mathrm{H}}=0,\\
&\Psi_{42}=0, && \Psi_{44}=0,\\
&\Delta\Psi_{42}=0, && \Delta\Psi_{44}=0.
\end{aligned} \tag{E-15}$$

For the multilayer orthorhombic medium considered above (see layer properties in Table 3 of part 1), the original effective parameters are

$$\begin{aligned}
&t_o=2.15879\,\mathrm{s},\\
&U_2=20.4326\,\mathrm{km^2/s},\quad W_{2x}=-1.58949\,\mathrm{km^2/s},\quad W_{2y}=-0.561274\,\mathrm{km^2/s},\\
&U_4=201.981\,\mathrm{km^4/s^3},\quad W_{42x}=-11.5086\,\mathrm{km^4/s^3},\quad W_{42y}=-0.701264\,\mathrm{km^4/s^3},\\
&\qquad W_{44x}=9.63248\,\mathrm{km^4/s^3},\quad W_{44y}=-1.96740\,\mathrm{km^4/s^3},
\end{aligned} \tag{E-16}$$

and the normalized effective parameters are

$$\begin{aligned}
&\bar{V}_2=3.07650\ \mathrm{km/s}, && e_2=8.24990\times10^{-2},\\
&V_{2,\mathrm{H}}=3.20089\ \mathrm{km/s}, && V_{2,\mathrm{L}}=2.94686\ \mathrm{km/s},\\
&V_{4,\mathrm{H}}=3.77553\ \mathrm{km/s}, && V_{4,\mathrm{L}}=3.67627\ \mathrm{km/s},\\
&e_{4,\mathrm{H}}=-1.91452\times10^{-2}, && e_{4,\mathrm{L}}=4.16582\times10^{-2},\\
&\bar{\eta}_{\mathrm{eff}}=0.136103, && \Psi_{2,\mathrm{H}}=-80.2755^\circ,\\
&\Psi_{42}=1.74350^\circ, && \Psi_{44}=-2.88592^\circ,\\
&\Delta\Psi_{42}=-7.98098^\circ, && \Delta\Psi_{44}=-12.6104^\circ.
\end{aligned} \tag{E-17}$$

Overall, the strength of the anisotropy of the effective model is governed by four effective parameters, $\bar{\eta}_{\mathrm{eff}}, e_2, e_{4,\mathrm{H}}, e_{4,\mathrm{L}}$, and the nonhyperbolic traveltime is affected by all of them. The azimuthal variation of the traveltime is governed by the strength of the second-order elliptic parameter e_2, the residual effective anellipticities $e_{4,\mathrm{H}}$ and $e_{4,\mathrm{L}}$, effective azimuth $\Psi_{2,\mathrm{H}}$, and two effective residual azimuths $\Delta\Psi_{42}$ and $\Delta\Psi_{44}$.

The validity range of the normalized effective parameters can be roughly estimated. The elliptic parameter e_2 is positive by definition, relating the high and low second-order NMO velocities to the azimuthally isotropic NMO velocity by equation E-2, and we assume its values are within the following range: $0\le e_2\le 0.5$. The effective azimuthally isotropic anellipticity may be in the range of $-0.2\le\bar{\eta}_{\mathrm{eff}}\le 0.6$, where the negative values are less likely (due to induced anellipticity). The high and low residual anellipticities have no induced component because they are defined as difference values in equation E-7. Therefore, negative and positive values are equally possible, and their range should be symmetric. We assume $|e_{4,\mathrm{H,L}}| \le 0.4$, but in most practical cases, the range will be smaller. The effective azimuth $\Psi_{2,\mathrm{H}}$ appears in all relationships with multiplicity $\omega_2 = 2$, and therefore its range is $0 \le \Psi_2 < \pi$. The effective residual azimuths $\Delta\Psi_{42}$ and $\Delta\Psi_{44}$ appear with multiplicities $\omega_{42} = 2$ and $\omega_{44} = 4$, and in addition, we choose the signs of W_{42} and W_{44} to keep the absolute values of residual azimuths minimum. From this, we conclude that the residual azimuths may be of any sign, and the range of their absolute values is $|\Delta\Psi_{42}| < \pi/4 = 45^\circ$ and $|\Delta\Psi_{44}| < \pi/8 = 22.5^\circ$.

APPENDIX F

FOURTH-ORDER NMO VELOCITY FOR MULTILAYER MODELS AND WEAK AZIMUTHAL ANISOTROPY

In this appendix, we consider a special case of weak azimuthal anisotropy of a layered orthorhombic medium. This case may be modeled by a VTI background with strong anisotropy (horizontal layering plane), and two mutually orthogonal vertical fracture planes, whose weakness components (Schoenberg and Helbig, 1997) are small. In this case, the two azimuthally isotropic parameters U_2 and U_4 can still be large, where the other six azimuthally anisotropic parameters $W_{2x}, W_{2y}, W_{42x}, W_{42y}, W_{44x}$, and W_{44y} are assumed small. We expand the second- and fourth-order NMO velocities into a power series of small effective parameters and drop the nonlinear terms with their products and powers. For weak azimuthal anisotropy, the second- and fourth-order NMO velocity functions (equation 18) and the effective anellipticity look alike in the slowness- and offset-azimuth domains:

$$\eta_{\mathrm{eff}}(\psi) = \frac{U_4 t_\mathrm{o}}{4U_2^2} - \frac{U_4 t_\mathrm{o}}{2U_2^3}(W_{2x}\cos 2\psi + W_{2y}\sin 2\psi) + \frac{t_\mathrm{o}}{4U_2^2}(W_{42x}\cos 2\psi + W_{42y}\sin 2\psi + W_{44x}\cos 4\psi + W_{44y}\sin 4\psi) - \frac{1}{8}, \tag{F-1}$$

where the azimuth ψ stands for either ψ_{slw} or ψ_{off}. We emphasize, however that even for weak azimuthal anisotropy, the slowness azimuth is not equal to the offset azimuth. Equations E-1 and E-16 may also be arranged with two additional fourth-order azimuths Ψ_{42} and Ψ_{44}:

$$\begin{aligned} V_2^2(\psi)t_\mathrm{o} &= U_2 + W_2\cos 2(\psi - \Psi_{2,\mathrm{H}}), \\ V_4^4(\psi)t_\mathrm{o} &= 2[U_4 + W_{42}\cos 2(\psi - \Psi_{42}) \\ &\quad + W_{44}\cos 4(\psi - \Psi_{44})], \end{aligned} \tag{F-2}$$

$$\eta_{\mathrm{eff}}(\psi) = \frac{t_\mathrm{o}}{4}\left\{\frac{U_4}{U_2^2} - \frac{2U_4}{U_2^3}W_2\cos 2(\psi - \Psi_{2,\mathrm{H}}) + \frac{1}{U_2^2}[W_{42}\cos 2(\psi - \Psi_{42}) + W_{44}\cos 4(\psi - \Psi_{44})]\right\} - \frac{1}{8}, \tag{F-3}$$

where

$$\begin{aligned} &W_2^2 = W_{2x}^2 + W_{2y}^2, &&\cos 2\Psi_{\mathrm{slow}} = \frac{W_{2x}}{W_2}, &&\sin 2\Psi_{\mathrm{slow}} = \frac{W_{2y}}{W_2}, \\ &W_{42}^2 = W_{42x}^2 + W_{42y}^2, &&\cos 2\Psi_{42} = \frac{W_{42x}}{W_{42}}, &&\sin 2\Psi_{42} = \frac{W_{42y}}{W_{42}}, \\ &W_{44}^2 = W_{44x}^2 + W_{44y}^2, &&\cos 4\Psi_{44} = \frac{W_{44x}}{W_{44}}, &&\sin 4\Psi_{44} = \frac{W_{44y}}{W_{44}}. \end{aligned} \tag{F-4}$$

Note that W_2 is a positive value, whereas W_{42} and W_{44} may be of any sign. In the case of weak azimuthal anisotropy, in terms of normalized global effective parameters, the second-and fourth-order NMO velocities and effective anellipticity become

$$\begin{aligned} V_2^2(\psi) &= \bar{V}_2^2[1 + e_2\cos 2(\psi - \Psi_{2,\mathrm{H}})], \\ V_4^4(\psi) &= \bar{V}_4^4 + \frac{V_{4,\mathrm{H}}^4 - V_{4,\mathrm{L}}^4}{2}\frac{\cos 2(\psi - \Psi_{42})}{\cos 2\Delta\Psi_{42}} + \frac{V_{4,\mathrm{H}}^4 - 2\bar{V}_4^4 + V_{4,\mathrm{L}}^4}{2}\frac{\cos 4(\psi - \Psi_{44})}{\cos 4\Delta\Psi_{44}}, \\ \eta_{\mathrm{eff}}(\psi) &= \bar{\eta}_{\mathrm{eff}} + e_2\frac{1 + 8\bar{\eta}_{\mathrm{eff}}}{4}\tan 2\Delta\Psi_{42}\sin 2(\psi - \Psi_{2,\mathrm{H}}) \\ &\quad + \frac{e_{4,\mathrm{H}} - e_{4,\mathrm{L}}}{2}\frac{\cos 2(\psi - \Psi_{42})}{\cos 2\Delta\Psi_{42}} + \frac{e_{4,\mathrm{H}} + e_{4,\mathrm{L}}}{2}\frac{\cos 4(\psi - \Psi_{44})}{\cos 4\Delta\Psi_{44}}, \end{aligned} \tag{F-5}$$

where, due to weak azimuthal anisotropy, the azimuth ψ is generic and stands for ψ_{slw} and ψ_{off}. Note that even when $e_{4,\mathrm{H}}$ and $e_{4,\mathrm{L}}$ vanish, for multilayer media, it is still true that $\eta_{\mathrm{eff}}(\psi) \ne \bar{\eta}_{\mathrm{eff}}$, except for azimuths $\Psi_{2,\mathrm{H}}$ and $\Psi_{2,\mathrm{H}} + \pi/2$. For single-layer media, the effective residual azimuths vanish, and the last relationship simplifies to

$$\eta_{\mathrm{eff}}(\psi) = \bar{\eta}_{\mathrm{eff}} + \frac{e_{4,\mathrm{H}} - e_{4,\mathrm{L}}}{2}\cos 2(\psi - \Psi_{2,\mathrm{H}}) + \frac{e_{4,\mathrm{H}} + e_{4,\mathrm{L}}}{2}\cos 4(\psi - \Psi_{2,\mathrm{H}}). \tag{F-6}$$

The lag between the offset and slowness azimuth for weak azimuthal anisotropy does not vanish:

$$\sin(\psi_{\mathrm{off}} - \psi_{\mathrm{slw}}) = -\frac{W_2}{U_2}\sin 2(\psi - \Psi_{2,\mathrm{H}}) + \left[\frac{U_4 W_2}{U_2^2}\sin 2(\psi - \Psi_{2,\mathrm{H}}) - \frac{W_{42}}{2U_2}\sin 2(\psi - \Psi_{42}) - \frac{W_{44}}{U_2}\sin 4(\psi - \Psi_{44})\right]p_h^2 + O(p_h^4), \tag{F-7}$$

where the expression in square brackets has the units of velocity squared. Note that even in the expression for the lag between the two azimuths, the azimuth on the right side is generic and may be either phase or offset azimuth. With the normalized effective parameters, equation F-7 becomes

$$\begin{aligned}\sin(\psi_{\text{off}}-\psi_{\text{slw}}) = -e_2 \sin 2(\psi-\Psi_{2,\text{H}}) \\ + \left\{ e_2 \frac{1+8\bar{\eta}_{\text{eff}}}{2} \sin 2(\psi-\Psi_{2,\text{H}}) \right. \\ - \left[(e_{4,\text{H}}-e_{4,\text{L}}) + e_2 \frac{1+8\bar{\eta}_{\text{eff}}}{2} \right] \\ \times \frac{\sin 2(\psi-\Psi_{42})}{\cos 2\Delta\Psi_{42}} - 2(e_{4,\text{L}}+e_{4,\text{H}}) \\ \left. \times \frac{\sin 4(\psi-\Psi_{44})}{\cos 4\Delta\Psi_{44}} \right\} \bar{V}_2^2 p_\text{h}^2. \end{aligned} \tag{F-8}$$

APPENDIX G

EFFECTIVE ANELLIPTICITY FOR A SINGLE LAYER AND WEAK ANISOTROPY

In this appendix, we refer to a single layer, in which all anisotropic parameters are considered weak. It follows from equations D-3 and D-4 of part 1 that for a single layer, there are no additional azimuths. All global effective azimuths are equal $\Psi_{2,\text{H}} = \Psi_{42} = \Psi_{44}$ and coincide with either ψ_{x_1} or ψ_{x_2} as one of the two vertical symmetry planes matches with $\Psi_{2,\text{H}}$. For weak anisotropy, the following relationships hold for the vertical-slowness coefficients for P-waves (see equations B-10–B-13 of part 1):

$$A = 1+2\delta_2, \quad B = 1+2\delta_1, \quad C = -2\eta_2, \quad D = -2\eta_1, \\ E = -2(\eta_1+\eta_2-\eta_3), \tag{G-1}$$

where the medium anellipticities are given by

$$\eta_1 = \frac{\varepsilon_1-\delta_1}{1+2\delta_1} \approx \varepsilon_1-\delta_1, \quad \eta_2 = \frac{\varepsilon_2-\delta_2}{1+2\delta_2} \approx \varepsilon_2-\delta_2, \\ \eta_3 = \frac{\varepsilon_1-\varepsilon_2-\delta_3(1+2\varepsilon_2)}{(1+2\varepsilon_2)(1+2\delta_3)} \approx \varepsilon_1-\varepsilon_2-\delta_3, \tag{G-2}$$

and $\delta_1, \delta_2, \delta_3, \varepsilon_1$, and ε_2 are Tsvankin's (1997) orthorhombic parameters.

For P-waves, the local effective parameters can be obtained from equations D-3 and D-4 of part 1

$$\left.\begin{aligned} u_2 &= (1+\delta_1+\delta_2) v_P^2 \Delta t_\text{o} \\ w_{2x} &= (\delta_2-\delta_1)\cos 2\psi_{x_1} v_P^2 \Delta t_\text{o} \\ w_{2y} &= (\delta_2-\delta_1)\sin 2\psi_{x_1} v_P^2 \Delta t_\text{o} \end{aligned}\right\} \text{second-order parameters},$$

$$\left.\begin{aligned} u_4 &= \tfrac{1+2(\delta_1+\delta_2)+4(\eta_1+\eta_2)-\eta_3}{2} v_P^4 \Delta t_\text{o} \\ w_{42x} &= -[2(\eta_1-\eta_2)+(\delta_1-\delta_2)]\cos 2\psi_{x_1} v_P^4 \Delta t_\text{o} \\ w_{42y} &= -[2(\eta_1-\eta_2)+(\delta_1-\delta_2)]\sin 2\psi_{x_1} v_P^4 \Delta t_\text{o} \\ w_{44x} &= \tfrac{\eta_3}{2}\cos 4\psi_{x_1} v_P^4 \Delta t_\text{o} \\ w_{44y} &= \tfrac{\eta_3}{2}\sin 4\psi_{x_1} v_P^4 \Delta t_\text{o} \end{aligned}\right\} \\ \text{fourth-order parameters}. \tag{G-3}$$

Note that the shear parameters f, γ_1, and γ_2 do not appear in equation F-2. This means that acoustic approximation is implicitly assumed with weak anisotropy. Using the effective parameters defined in equation F-2, we obtain the effective anellipticity function, and it coincides with the results obtained by Pech and Tsvankin (2004)

$$\begin{aligned}\eta_{\text{eff}}(\psi) &= \eta_1 \sin^2(\psi-\psi_{x_1}) + \eta_2 \cos^2(\psi-\psi_{x_1}) \\ &\quad - \eta_3 \cos^2(\psi-\psi_{x_1})\sin^2(\psi-\psi_{x_1}), \\ &= \frac{4(\eta_1+\eta_2)-\eta_3}{8} - \frac{\eta_1-\eta_2}{2}\cos 2(\psi-\psi_{x_1}) \\ &\quad + \frac{\eta_3}{4}\cos 4(\psi-\psi_{x_1}). \end{aligned} \tag{G-4}$$

APPENDIX H

QUARTIC COEFFICIENT FOR A SINGLE ELASTIC LAYER WITH STRONG ANISOTROPY

In this appendix, we compute the quartic coefficient in the offset-azimuth domain for a single orthorhombic layer, and we compare it with the result obtained by Al-Dajani et al. (1998). In that paper, the moveout is presented by a nonnormalized quartic term with coefficient A_4' as

$$t^2 = t_\text{o}^2 + A_2 h^2 + \frac{A_4' h^2}{1+A_C h^2}, \tag{H-1}$$

where A_C is the asymptotic correction factor and A_4' is related to our normalized quartic term A_4 by

$$A_4' = \frac{A_4}{v_2^2 t_\text{o}^2} = -\frac{v_4^4-v_2^4}{4v_2^4} \cdot \frac{1}{v_2^2 t_\text{o}^2} = -\frac{v_4^4-v_2^4}{4v_2^6 t_\text{o}^2}. \tag{H-2}$$

The fourth-order NMO velocity is computed using equations 11, 12, and D-3 to D-7. For a medium consisting of a single orthorhombic layer, the local parameters are also global. In this case, the fourth-order parameters are not all independent, and the following constraint holds:

$$\begin{aligned} w_{2x} &= w_2 \cos 2\psi_{x_1}, & w_{2y} &= w_2 \sin 2\psi_{x_1}, \\ w_{42x} &= w_{42} \cos 2\psi_{x_1}, & w_{42y} &= w_{42} \sin 2\psi_{x_1}, \\ w_{44x} &= w_{44} \cos 4\psi_{x_1}, & w_{44y} &= w_{44} \sin 4\psi_{x_1}. \end{aligned} \tag{H-3}$$

With this in mind, the nonnormalized quartic coefficient for a single layer may be presented as

$$\begin{aligned} A_4' &= A_{4,0} + A_{4,2}\cos 2(\psi_{\text{off}}-\psi_{x_1}) \\ &\quad + A_{4,4}\cos 4(\psi_{\text{off}}-\psi_{x_1}), \end{aligned} \tag{H-4}$$

where

$$A_{4,0} = \frac{2u_2^6 - 4t_o u_4 w_2^4 + w_2^6 - u_2^4(4t_o u_4 + 3w_2^2)}{8(u_2^2 - w_2^2)^4} - \frac{3t_o u_2 w_2 w_{42}(u_2^2 + w_2^2) + 2t_o u_2^2 w_2^2 (2u_4 + 3w_{44})}{2(u_2^2 - w_2^2)^4},$$
$$A_{4,2} = \frac{u_2^5 w_2 - t_o w_{42}(u_2^4 + 6u_2^2 w_2^2 + w_2^4)}{2(u_2^2 - w_2^2)^4} - \frac{u_2 w_2^3(2u_2^2 - w_2^2) + 4t_o u_2 w_2 (u_2^2 + w_2^2)(u_4 + w_{44})}{2(u_2^2 - w_2^2)^4},$$
$$A_{4,4} = \frac{w_2^2(u_2^4 + w_2^4 - 2u_2^2 w_2^2 - 8t_o u_2^2 u_4)}{8(u_2^2 - w_2^2)^4} - \frac{t_o u_2 w_2 w_{42}(u_2^2 + w_2^2) + 4t_o w_{44}(u_2^4 + w_2^4)}{2(u_2^2 - w_2^2)^4}. \quad \text{(H-5)}$$

Next, we rearrange equation H-4 as

$$\begin{aligned} A_4' &= (A_{4,0} + A_{4,2} + A_{4,4})\cos^2(\psi_{\text{off}} - \psi_{x_1}) \\ &\quad + (A_{4,0} - A_{4,2} + A_{4,4})\sin^2(\psi_{\text{off}} - \psi_{x_1}) \\ &\quad - 8A_{4,4}\cos^2(\psi_{\text{off}} - \psi_{x_1})\sin^2(\psi_{\text{off}} - \psi_{x_1}) \\ &= (A_{4,0} + A_{4,2} + A_{4,4})\cos^4(\psi_{\text{off}} - \psi_{x_1}) \\ &\quad + (A_{4,0} - A_{4,2} + A_{4,4})\sin^4(\psi_{\text{off}} - \psi_{x_1}) \\ &\quad + 2(A_{4,0} - 3A_{4,4})\cos^2(\psi_{\text{off}} - \psi_{x_1})\sin^2(\psi_{\text{off}} - \psi_{x_1}). \end{aligned} \quad \text{(H-6)}$$

We introduce new notations

$$B_C = A_{4,0} + A_{4,2} + A_{4,4},$$
$$B_S = A_{4,0} - A_{4,2} + A_{4,4},$$
$$B_M = 2(A_{4,0} - 3A_{4,4}), \quad \text{(H-7)}$$

and equation H-6 becomes

$$A_4' = B_C\cos^4(\psi_{\text{off}} - \psi_{x_1}) + B_S\sin^4(\psi_{\text{off}} - \psi_{x_1}) + B_M\cos^2(\psi_{\text{off}} - \psi_{x_1})\sin^2(\psi_{\text{off}} - \psi_{x_1}), \quad \text{(H-8)}$$

where the coefficients are obtained by combining equations H-5 and H-7

$$B_C = \frac{(u_2 - w_2)^2 - 2t_o(u_4 + w_{42} + w_{44})}{4(u_2 - w_2)^4},$$
$$B_S = \frac{(u_2 + w_2)^2 - 2t_o(u_4 - w_{42} + w_{44})}{4(u_2 + w_2)^4},$$
$$B_M = \frac{u_2^2 - w_2^2 - 2t_o(u_4 - 3w_{44})}{2(u_2 - w_2)^2(u_2 + w_2)^2}. \quad \text{(H-9)}$$

Local effective parameters are expressed through the coefficients $A, \ldots, E$ of the vertical-slowness surface in equations D-3 and D-4 of part 1

$$u_2 = \frac{A + B}{2} v_{\text{ver}}^2 \Delta t_o,$$
$$w_2 = \frac{A - B}{2} v_{\text{ver}}^2 \Delta t_o,$$
$$u_4 = \left(\frac{3A^2 + 2AB + 3B^2}{16} - \frac{3C + 3D + E}{4}\right) v_{\text{ver}}^4 \Delta t_o,$$
$$w_{42} = \left(\frac{A^2 - B^2}{4} - C + D\right) v_{\text{ver}}^4 \Delta t_o,$$
$$w_{44} = \left[\frac{(A - B)^2}{16} - \frac{C + D - E_i}{4}\right] v_{\text{ver}}^4 \Delta t_o. \quad \text{(H-10)}$$

Introduction of these equations into equation H-9 results in

$$B_C = \frac{C}{A^4 v_{\text{ver}}^4 t_o^2}, \quad B_S = \frac{D}{B^4 v_{\text{ver}}^4 t_o^2}, \quad B_M = \frac{E}{A^2 B^2 v_{\text{ver}}^4 t_o^2}. \quad \text{(H-11)}$$

So far, equation H-11 is valid for any wave type. Next, we apply the coefficients $A, \ldots, E$ for P-waves to compare our solution with Al-Dajani's. Recall that for P-wave $v_{\text{ver}} = v_P$, and from equation B-10 of part 1

$$A = 1 + 2\delta_2, B = 1 + 2\delta_1. \quad \text{(H-12)}$$

Al-Dajani et al. (1998) use slightly different parameterization, applying $f_1$andf_2, where f_2 corresponds to our standard parameter f

$$f_1 = \frac{C_{33} - C_{44}}{C_{33}} = 1 - v_P^2/v_{S,x_2}^2,$$
$$f_2 = \frac{C_{33} - C_{55}}{C_{33}} = 1 - v_P^2/v_{S,x_1}^2, \quad \text{(H-13)}$$

where v_{S,x_1} and $v_{S,x_2}$are the vertical shear velocities polarized in the directions x_1 and x_2, respectively. The system becomes redundant, and equation A-8 of part 1 yields the following constraint:

$$(1 - f_1)(1 + 2\gamma_2) = (1 - f_2)(1 + 2\gamma_1) = \frac{C_{66}}{C_{33}}. \quad \text{(H-14)}$$

Al-Dajani et al. (1998) do not use parameter γ_1; they use f_1, f_2, γ_2. Actually, one can use all parameters of the redundant set, provided constraint H-14 is acknowledged. With the definitions of equation H-13, equations B-11 and B-12 of part 1 result in

$$C = -\frac{2(\varepsilon_2 - \delta_2)(f_2 + 2\delta_2)}{f_2},$$
$$D = -\frac{2(\varepsilon_1 - \delta_1)(f_1 + 2\delta_1)}{f_1}. \quad \text{(H-15)}$$

Introducing equations H-12 and H-15 into the first two equations of equation set H-11 results in

$$
B_C = -\frac{2(\varepsilon_2 - \delta_2)(f_2 + 2\delta_2)}{(1+2\delta_2)^4 f_2 v_P^4 t_o^2},
$$
$$
B_S = -\frac{2(\varepsilon_1 - \delta_1)(f_1 + 2\delta_1)}{(1+2\delta_1)^4 f_1 v_P^4 t_o^2}, \tag{H-16}
$$

which coincides with the results obtained by Al-Dajani et al. (1998, equations 10 and 11). Next, we show that parameter B_M in the third equation of equation set H-11 also matches the results of the cited paper. It follows from the third equation of equation set H-11 that the denominator of B_M is identical to this of Al-Dajani et al. (1998), so we compare the numerator. Al-Dajani's result in equation 12 of their paper has factor two in the numerator, and we rearrange parameter E, which is given in equation B-13 of part 1 as

$$
E = 2(Q + K), \tag{H-17}
$$

where

$$
Q = \frac{(C_{13}+C_{55})^2(C_{44}-C_{66})}{2C_{33}(C_{33}-C_{55})^2} + \frac{(C_{23}+C_{44})^2(C_{55}-C_{66})}{2C_{33}(C_{33}-C_{44})^2} + \frac{(C_{13}+C_{55})^2(C_{23}+C_{44})^2(2C_{33}-C_{44}-C_{55})}{2C_{33}(C_{33}-C_{44})^2(C_{33}-C_{55})^2}, \tag{H-18}
$$

and

$$
K = -\frac{(C_{13}+C_{55})(C_{23}+C_{44})(C_{12}+C_{66})}{C_{33}(C_{33}-C_{44})(C_{33}-C_{55})} = -\frac{A_{12}A_{13}A_{23}}{C_{33}^3 f_1 f_2}, \tag{H-19}
$$

where parameters A_{12}, A_{13}, and A_{23} are defined in equation A-6 of part 1. It follows from equations A-9–A-11 of part 1 that:

$$
\frac{A_{13}}{C_{33}} = \sqrt{f_2(f_2+2\delta_2)}, \frac{A_{23}}{C_{33}} = \sqrt{f_1(f_1+2\delta_1)},
$$
$$
\frac{A_{12}}{C_{33}} = \sqrt{2\delta_3(1+2\varepsilon_2)[(1+2\varepsilon_2)-(1+2\gamma_2)(1-f_1)]+[(1+2\varepsilon_2)-(1+2\gamma_2)(1-f_1)]^2}. \tag{H-20}
$$

Converting the stiffness components into Tsvankin's (1997) orthorhombic parameters, we obtain

$$
Q = \frac{(f_1+2\delta_1)[\delta_2(f_1+f_2)+f_1 f_2]}{f_1 f_2} - \frac{2(1-f_1)(f_1\delta_2+f_2\delta_1+f_1 f_2)\gamma_2}{f_1 f_2}, \tag{H-21}
$$

$$
K = -[(1+2\gamma_2)f_1 + 2(\varepsilon_2-\gamma_2)]\sqrt{\frac{f_1+2\delta_1}{f_1}\cdot\frac{f_2+2\delta_2}{f_2}} \times \sqrt{1+\frac{2\delta_3(1+2\varepsilon_2)}{(1+2\gamma_2)f_1+2(\varepsilon_2-\gamma_2)}}. \tag{H-22}
$$

The expression in the square brackets in Al-Dajani's et al. (1998) equation 12 includes five terms. The sum of the first four terms is identical to our parameter Q (although these terms are presented there in a slightly different form), and the last term there coincides with our parameter K. In the case of acoustic approximation, $f_1 = f_2 = 1$, and equations H-21 and H-22 simplify to

$$
Q = (1+2\delta_1)(1+2\delta_2),
$$
$$
K = -\sqrt{1+2\delta_1}\sqrt{1+2\delta_2}\sqrt{1+2\delta_3}(1+2\varepsilon_2),
$$
$$
E = 2(Q+K) = 2(1+2\delta_1)(1+2\delta_2)\left[1-\sqrt{\frac{(1+2\eta_1)(1+2\eta_2)}{1+2\eta_3}}\right], \tag{H-23}
$$

where η_1, η_2, and η_3 are the intrinsic anellipticities defined in equation G-2.

LIST OF SYMBOLS

Interval parameters

v_P	=	vertical compressional velocity of the orthorhombic layer
v_{S,x_1}	=	vertical velocity of S-wave polarized in thex_1 direction
v_{S,x_2}	=	vertical velocity of S-wave polarized in the x_2 direction
$f = f_2$	=	coefficient depending on the ratio between vertical shear velocity, polarized in the x_1 direction, and the vertical compressional velocity
f_1	=	coefficient depending on the ratio between vertical shear velocity, polarized in the x_2 direction, and vertical compressional velocity (redundant but suitable parameter)
$\delta_1, \delta_2, \delta_3, \varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2$	=	Tsvankin's (1997) orthorhombic parameters of a layer
η_1, η_2, η_3	=	intrinsic anellipticities of orthorhombic media

Local parameters

$\Delta t_{o,i}$	=	local vertical time (two way if not specified otherwise)
$v_2(\psi)$	=	azimuthally dependent local second-order NMO velocity
$v_4(\psi)$	=	azimuthally dependent local fourth-order NMO velocity
$\eta_{\mathrm{eff}}^{\mathrm{loc}}(\psi)$	=	azimuthally dependent local effective anellipticity
$u_{2,i}, w_{2x,i}, w_{2y,i}$	=	local second-order effective parameters; i is the layer number
$u_{4,i}, w_{42x,i}, w_{42y,i}, w_{44x,i}, w_{44y,i}$	=	local fourth-order effective parameters; i is the layer number
A, B, C, D, E	=	series coefficients of the vertical slowness

Global parameters

p_h	=	magnitude of the horizontal-slowness vector
h_R	=	radial offset component

h_T = transverse offset component
h = offset magnitude
$\bar{h}$ = normalized offset
t_o = two-way vertical time
t = two-way traveltime
$\bar{t}$ = normalized two-way traveltime
z = depth of reflector point
V_{ave} = global average vertical velocity
$V_{2,\text{L}}$ = global effective low second-order NMO velocity
$V_{2,\text{H}}$ = global effective high second-order NMO velocity
$\overline{V}_2$ = azimuthally isotropic second-order NMO velocity
e_2 = effective elliptic parameter
$V_{4,\text{L}}$ = global effective fourth-order NMO velocity computed at the azimuth of the global effective low second-order NMO velocity
$V_{4,\text{H}}$ = global effective fourth-order NMO velocity computed at the azimuth of the global effective high second-order NMO velocity
$\overline{V}_4$ = azimuthally isotropic fourth-order NMO velocity
$\eta_{\text{eff,L}}$ = effective anellipticity computed at the azimuth of the global effective low second-order NMO velocity
$\eta_{\text{eff,H}}$ = effective anellipticity computed at the azimuth of the global effective high second-order NMO velocity
$\bar{\eta}_{\text{eff}}$ = azimuthally isotropic effective anellipticity
$e_{4,\text{L}}$ = residual effective anellipticity computed at the azimuth of the global effective low second-order NMO velocity
$e_{4,\text{H}}$ = residual effective anellipticity computed at the azimuth of the global effective high second-order NMO velocity
α = normalized asymptotic correction factor of the traveltime approximation
A_C = nonnormalized asymptotic correction factor of the traveltime approximation
$V_2(\psi)$ = generic azimuthally dependent global second-order NMO velocity
$V_4(\psi)$ = generic azimuthally dependent global fourth-order NMO velocity
$V_{2,\text{slw}}(\psi_{\text{slw}})$ = slowness-azimuth/slowness domain global second-order NMO velocity
$V_{4,\text{slw}}(\psi_{\text{slw}})$ = slowness-azimuth/slowness domain global fourth-order NMO velocity
$V_{2,\text{off}}(\psi_{\text{slw}})$ = slowness-azimuth/offset domain global second-order NMO velocity
$V_{4,\text{off}}(\psi_{\text{slw}})$ = slowness-azimuth/offset domain global fourth-order NMO velocity
$V_{2,\text{slw}}(\psi_{\text{off}})$ = offset-azimuth/slowness domain global second-order NMO velocity
$V_{4,\text{slw}}(\psi_{\text{off}})$ = offset-azimuth/slowness domain global fourth-order NMO velocity
$V_{2,\text{off}}(\psi_{\text{off}})$ = offset-azimuth/offset domain global second-order NMO velocity
$V_{4,\text{off}}(\psi_{\text{off}})$ = offset-azimuth/offset domain global fourth-order NMO velocity
$K_{2,\text{slw}}(\psi_{\text{off}})$ = offset-azimuth/slowness domain normalizing factor of fourth-order NMO velocity
$K_{4,\text{slw}}(\psi_{\text{off}})$ = offset-azimuth/slowness domain kernel of fourth-order NMO velocity
$K_{2,\text{off}}(\psi_{\text{off}})$ = offset-azimuth/offset domain normalizing factor of fourth-order NMO velocity
$K_{4,\text{off}}(\psi_{\text{off}})$ = offset-azimuth/offset domain kernel of fourth-order NMO velocity
$\mathbf{m}$ = row vector of length five with global high-order parameters
$\mathbf{a}_{\text{slw},5}(\psi_{\text{slw}})$ = vector of length five dependent on slowness azimuth only
$\mathbf{a}_{\text{slw},7}(\psi_{\text{slw}})$ = vector of length seven dependent on slowness azimuth only
$\mathbf{a}_{\text{off},5}(\psi_{\text{off}})$ = vector of length five dependent on offset azimuth only
$\mathbf{a}_{\text{off},7}(\psi_{\text{off}})$ = vector of length seven dependent on offset azimuth only
$\mathbf{M}_{\text{slw}}^{\text{lag}}$ = matrix needed to compute the azimuthal lag in slowness-azimuth domain
$\mathbf{M}_{\text{off}}^{\text{B}}$ = matrix needed to compute the slowness azimuth versus the offset azimuth
$\mathbf{M}_{\text{slw/off}}$ = matrix needed to compute the fourth-order NMO velocity in the slowness-azimuth/offset domain
$\mathbf{M}_{\text{off/slw}}$ = matrix needed to compute the fourth-order NMO velocity in the offset-azimuth/slowness domain
$\mathbf{M}_{\text{off/off}}$ = matrix needed to compute the fourth-order NMO velocity in the offset-azimuth/offset domain
$\eta_{\text{eff}}(\psi)$ = azimuthally dependent global effective anellipticity in slowness- or offset-azimuth domain
$A_4(\psi)$ = normalized quartic coefficient of moveout in slowness- or offset-azimuth domain
$A_4'(\psi)$ = nonnormalized quartic coefficient of moveout in slowness- or offset-azimuth domain
U_2, W_{2x}, W_{2y} = global second-order effective parameters
$U_4, W_{42x}, W_{42y}, W_{44x}, W_{44y}$ = global fourth-order effective parameters

Azimuths

ψ_{x_1} = azimuth of local orthorhombic axis x_1
ψ_{x_2} = azimuth of local orthorhombic axis x_2
ψ_{slw} = slowness azimuth, phase-velocity azimuth
ψ_{off} = offset azimuth (acquisition azimuth)
ψ = generic azimuth, may be either slowness azimuth or offset azimuth
$\Psi_{2,\text{L}}$ = azimuth of global low second-order NMO velocity
$\Psi_{2,\text{H}}$ = azimuth of global high second-order NMO velocity
Ψ_{42} = fourth-order global effective azimuth related to basic azimuthal multiplicity
Ψ_{44} = fourth-order global effective azimuth related to doubled azimuthal multiplicity
$\Delta\Psi_{42}$ = fourth-order global effective residual azimuth related to basic azimuthal multiplicity

$\Delta\Psi_{44}$	=	fourth-order global effective residual azimuth related to doubled azimuthal multiplicity
$\omega_2 = 2$	=	azimuthal multiplicity of second-order terms
$\omega_{42} = 2$	=	basic azimuthal multiplicity of fourth-order terms
$\omega_{44} = 4$	=	doubled azimuthal multiplicity of fourth-order terms

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