An acceleration method for the anti-leakage parabolic Radon transform for seismic data interpolation
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SUMMARY
An approach to accelerate performance of the parabolic Radon anti-leakage method for multidimensional seismic interpolation is presented. The anti-leakage algorithm aims to reduce the leakage of energy between the components chosen to approximate the data, and that build the spatial spectrum. Reducing leakage is traditionally done by an iterative process of projections and back projections between the spatial and transformed domains. In this work we show a way to avoid the need for the repeating compute intensive transform computations for the Radon based anti-leakage algorithm. This improvement is based on similar ideas presented for the Fourier based anti-leakage algorithm, and shows a substantial runtime improvement of a factor of 6 compared to the old approach.

INTRODUCTION
Multidimensional regularization of seismic data is an important practice in exploration seismology. Various data processing methods, such as wave-equation imaging and multiple-elimination, require seismic data to be regularly sampled. In reality, field acquisition patterns are most often non-regular or coarse due to the presence of natural or man made obstacles on land, and in marine surveys, due to wells and cable drift caused by wind and waves. Seismic interpolation is hence used in those cases to regularize the data. Seismic interpolation is also used for filling holes in the data to improve sampling for migration and for AVO analysis.

One of the methods for 5D Fourier based seismic interpolation is the Anti-leakage Fourier Transform (ALFT) (Xu and Pham, 2004; Xu et al., 2010). The main goal of the ALFT method is to reduce the leakage between wave-numbers arising from irregularities in the spatial sampling of the input 5D gather. The algorithm is quite straightforward, and involves iterative calculations of irregular forward and backward discrete Fourier transforms (DFTs). In every iteration a maximal energy coefficient is chosen, and its back projection into the spatial domain is subtracted from the input data. The chosen coefficient is added to the coefficients picked on previous iterations, and the process repeats until convergence.

The ALFT method was originally formulated to work with the Fourier transform in the f – k domain. A natural extension to this method would be to use the same approach for the parabolic Radon transform in the f – p domain. Finding a sparse representation to the data using Radon coefficients would be complementary to other sparse inversion methods in the Radon space (Trad et al., 2002, 2003; Wang and Nimsaila, 2014).

The iterative nature of anti-leakage transform methods burdens on their performance as a result of the repeating forward and backward transforms required at each step. The Fourier based method requires the computation of forward and backward irregular DFTs, which generally have an \( \mathcal{O}(N_sN_p) \) complexity (with \( N_s \) as the number of spatial samples and \( N_p \) as the number of spatial frequencies). Several efficient algorithms to calculate the irregular DFT have been proposed in the past (e.g. Fournier (2003)), but these speed those algorithms only to a limited extent.

For the Fourier based anti-leakage methods (i.e. ALFT (Xu and Pham, 2004), and OMP (Hollander et al., 2012)), a method to accelerate performance was presented by Whiteside et al. (2014); Jahanjooy et al. (2016); and Hollander et al. (2017). The method is based on pre-calculating the spectral leakage pattern between Fourier functions sampled irregularly is space. The spectral leakage pattern depends only on geometry, and is calculated once. It is then used for all frequency slices that are inverted in the ALFT algorithm is such a way that allows the residual at each iteration to be updated by the new picked coefficient without the need to go back and forth between the spatial and wave-number domains. In this paper an expansion of the method is presented for an anti-leakage transform for data regularization which is based on the parabolic Radon transform.

METHODS
We start by representing the seismic gather \( \mathbf{d}(t, \mathbf{x}) \) as a combination of multidimensional Radon basis functions. This approximation can be written in the most general form as follows

\[
\mathbf{d}(t, \mathbf{x}) \approx \sum_j m_j e^{i2\pi t f_j} e^{i2\pi f x_j},
\]

with \( t \) and \( f \) denoting the vertical regularly sampled dimension in the time and frequency domains, respectively. Following a preliminary vertical FFT to separate the gather to temporal frequency slices, each slice is approximated as

\[
\mathbf{d}(f, \mathbf{x}) \approx \sum_j m_j e^{i2\pi f x_j^2} p_j = Am,
\]

where \( \mathbf{m} \) is the vector of Radon coefficients for a particular frequency \( f \). The matrix \( \mathbf{A} \) contains in its columns the corresponding Fourier basis functions sampled at the irregular 4D spatial grid \( \mathbf{x} \), scaled for Radon by itself and by the temporal frequency \( f (x := fx^2) \).

The 4D wave number vector which is denoted by \( \mathbf{p} \) in both (1) and (2) represents the coefficient of the quadratic term of the parabolas that compose the transform.

The Anti-leakage Parabolic Radon Transform (ALPRT) method
The ALPRT is a matching pursuit approach to find a sparse representation of the form (2) in the sense that only a handful of the available basis functions are chosen to participate in
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the final expansion. As shown by others (Wang et al., 2010; Schonewille et al., 2014; Cao and Ross, 2017), the algorithm is iterative in nature; at each step and for each frequency slice the most dominant coefficient is chosen. Its projection back to the spatial domain is subtracted from the original data, to form a residual from which the most dominant coefficient is picked in the next iteration. In this work we present a variant of the original slice-by-slice inversion that relates coefficients from different frequency slices to guide the coefficients picked for all slices in the next iteration. The rationale for this approach is that the parabolic Radon transform collapses parabolic events in the time-spatial domain \((t-x)\) to vertical lines in the frequency wave-number domain \((f-p)\). This allows a convenient guidance of coefficient picking that belong to the same coherent event between the different slices. The ALPRT steps are described therefore as follows:

Initialization:

1. Transform the seismic 5D volume \(\hat{d}(t,x)\) vertically from \(t-x\) to \(f-x\) to form \(d(f,x)\).
2. For each temporal frequency slice in \(d(f,x)\), scale the irregular geometry vector \(x\) by itself and by the temporal frequency \(f(x := fx^2)\), and calculate its spatial discrete Fourier transform, to form the volume \(\psi^{(i)} = \text{DFT}(d)\).

Iterate over \(i\), starting at \(i=1\):

1. Stack the volume \(\|v^{(i-1)}\|\) in the vertical direction to obtain a masking slice \(w(p)\).
2. For each frequency slice update the coefficient vector \(m_f\). Weight the frequency slice \(v^{(i-1)}\) taken from the volume \(v^{(i-1)}\) by \(w\). Find the dominant coefficient in the weighted slice \(w \cdot v^{(i-1)}\) and add its corresponding value in \(v^{(i-1)}\) to the existing coefficient vector (we regard \(m^{(0)}\) as an empty vector)
   \[
   m_f^{(i)} = m_f^{(i-1)} + \frac{m_f^{(i-1)} \cdot v_f^{(i-1)}}{\max_p |w \cdot v_f^{(i-1)}|}.
   \]
3. Calculate for each slice the vector of Fourier coefficients of the residual,
   \[
   \psi^{(0)} = \text{DFT} \left[ r^{(0)} \right] = \text{DFT} \left[ d_f - \text{DFT}^{-1} \left( m_f^{(0)} \right) \right],
   \]
   where DFT is the discrete Fourier transform operator (and \(\text{DFT}^{-1}\) is its backward projection), and the subscript \((\bullet)_f\) stands for the frequency slice \(f\) taken from the relevant volume.
4. Terminate iterations for a particular frequency slice if
   \[
   \frac{\|\psi^{(i)}\|_2}{\|\psi^{(0)}\|_2} < \varepsilon,
   \]
   where \(\|\psi^{(0)}\|_2\) is the maximal \(l_2\) norm of all slices in the 5D volume \(\psi^{(0)}\), and \(\varepsilon\) is a given convergence threshold.

The most time consuming step in the algorithm described above is number 3, in which the residual \(r^{(i)}\) is formed by a backward transform of the vector of coefficients picked up to a certain iteration, and then forward transformed to form \(v_f^{(i)}\), from which the next dominant coefficient is picked in the next iteration. In the next section we describe an approach that substantially improves the performance for the ALPRT algorithm by the pre-calculation of the temporal frequency dependent spectral leakage patterns between the transform underlying Fourier basis functions.

Pre-calculation of geometry dependent spectral leakage patterns to accelerate performance

The method to accelerate performance for the ALPRT algorithm is based on earlier works by Whiteside et al. (2014) and Jahanjooy et al. (2016). We can rewrite equation (4) in a matrix notation form as

\[
\psi_f^{(i)} = F_f^* \left( d_f - A_f^{(i)} m_f^{(i)} \right) = \psi_f^{(0)} - F_f^* A_f^{(i)} m_f^{(i)}.
\]

The matrix \(F_f\) in (6) is a dictionary of Fourier functions for a particular frequency slice \(f\), which are sampled over the 4D spatial geometry, scaled for Radon by \(x := fx^2\). The matrix \(A_f^{(i)}\) contains in its columns the Fourier functions, sampled over the scaled geometry, of the coefficients that were chosen up to a certain iteration \(i\). The Hermitian transpose operator is denoted by \(\left(\bullet\right)^*\).

The multiplication \(F_f^* A_f^{(i)}\) in (6) represents the leakage pattern between the Fourier functions associated with the coefficients picked up to a certain iteration, and all other Fourier functions in the dictionary. This matrix is a sub-matrix of the full \(F_f^* F_f\), which represents the leakage of all Fourier functions that can potentially be picked in the iterative process. It can readily be shown that the matrix \(F_f^* F_f\) has a Toeplitz structure. This observation means it is possible to calculate its first column by the forward transform of a Dirac Comb, and that would be sufficient to obtain all of its possible values.

Based on the above observation the following steps are added to the ALPRT algorithm described above:

Initialization (added step): Calculate a volume of leakage patterns. For each frequency slice forward transform a Dirac Comb sampled on the geometry \(x\), scaled for Radon by \(x := fx^2\).

In the iterative process replace equation (4) by (6). During an iteration \(i\) for a particular frequency \(f\) add a column to the matrix \(F_f^* A_f^{(i)}\) by choosing the proper elements from the leakage pattern volume that was pre-calculated during initialization (the fist column in \(F_f^* F_f\) for each \(f\)). Multiply the obtained matrix by the vector of coefficients at the current iteration, and update the slice \(\psi_f^{(i)}\) for the next iteration.

The core of this improvement, and the fact it results in a better algorithmic efficiency, is due to the fact that the update to the transform residual volume \(\psi^{(i)}\) is done for each \(f\) only in the spatially transformed domain \(p\), without going back and
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Figure 1: (a) An input aperture composed of 15 cdp gathers. (b) The middle input cdp is regularized to a rectangular equispaced offset pattern. On the left gather in (b) reconstruction was made with the original algorithm, and on the middle reconstruction was done using the accelerated approach. Computation time is given in seconds for each reconstruction. The negligible differences between the results are shown on the right gather.

Figure 2: Location of shots (red dots) and receivers (blue dots) for the input geometry (a) and output regularized geometry (b)
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for the original spatial domain. Note that unlike a similar improvement for ALFT, the leakage patterns for the Radon transform are different for each frequency slice \( f \), and hence stored in memory as leakage pattern volumes. This is required because, like explained above, for the Radon the algorithm traverses through all frequency slices in a particular iteration, and uses the entire residual volume in the transformed domain to direct the picking of coefficients in the next iteration.

**EXAMPLES**

An input super-gather of 15 cdp’s (5 in the inline and 3 in the crossline directions) was used as input to the 5D ALPRT algorithm (figure 1a). The input geometry of the middle aperture gather is plotted in figure 2a, which shows the distribution of shots and receivers. The middle cdp was reconstructed in figure 1b to a rectangular pattern with an equal distribution of offsets in both directions (figure 2b). Reconstruction in figure 1b was done using the original algorithm, and the improved version accelerated for performance. A comparison between the outputs shows negligible differences between the results (right gather in figure 1b).

The run to reconstruct the gather in figure 1 was done with a dictionary size of 100 parabolas, out of which the algorithm picked exactly 30 coefficients for each frequency slice. The runtime in seconds for each run is printed in figure 1b for each reconstruction. As can be seen for this particular tested case the improved algorithm concludes the run about 6 time faster than the original one without the improvement.

The effect of two calculation parameters on runtime differences between the methods was tested. Figure 3 shows the effect the total number of coefficients picked for each frequency slice has on the overall runtime. As can be seen the new accelerated approach is less sensitive to the number of coefficients chosen. In figure 4 the effect of the dictionary size on total runtime is given. Here both approaches are sensitive to the size of the dictionary, but the new approach is still consistently faster than the original algorithm.

**DISCUSSION**

Fourier based anti-leakage regularization methods for seismic data are very commonly used in the industry. These methods suffer greatly from performance due to the need to repeatedly go back and forth between the spatial and transform domains, by way of reducing the spectral leakage between the coefficients chosen to represent the data. Several approaches were presented in the past to accelerate ALFT by performing residual updates solely in the transform domain.

In this work we expand the approach to an anti-leakage transform method which is based on the parabolic Radon. The parabolic Radon represents seismic data better than the sine/cosine expansion and this allows for a sparser domain. This transform also handles aliasing better than Fourier. However, the same performance issues tackled by ALFT arise in the Radon based method. The implementation of the accelerating approach for Radon presented in this work is different than that for Fourier because of the need to calculate leakage pattern volumes, in which each leakage pattern slice is computed by scaling the geometry by a different frequency.

A substantial improvement in performance is demonstrated for a representative case. The improvement is less sensitive to the number of coefficients the algorithm chooses for each frequency slice, but more sensitive to the size of dictionary of Radon basis functions available in the analysis. These results are probably related with the fact that calculating the leakage volume depends on the size of the dictionary. But once the patterns are established, updating the residual in the iterative process involves less number of operations that depend heavily on the size of the dictionary.

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REFERENCES


