Eigenray method: Geometric spreading

Zvi Koren* and Igor Ravve, Emerson

Summary

At the last SEG meeting we presented the Eigenray method for finding stationary rays in 3D smooth heterogeneous general anisotropic media by applying a non-linear finite element scheme. Starting with an initial path, the convergence to the stationary ray is based on the Newton method that requires both gradient and Hessian of the traveltime with respect to nodal locations and ray velocity directions. In this work we show how the global Hessian matrix of the stationary ray can be used to compute and geometric spreading. We compare the accuracy of geometric spreading computed by our proposed Eigenray method with analytic solutions for two benchmark examples.

Introduction

Geometric spreading is the principal component of the Green’s function, accounting for amplitude decay along the propagating waves/rays. There is a great wealth of published material available on this topic (e.g., Červený, 2001; Schleicher et al., 2007; Tsvankin and Grechka, 2011, and many others). It is an important characteristic for both seismic forward modeling and seismic migration/inversion. In particular, geometric spreading is applied as one of the seismic migration weighting components in the Kirchhoff integral to obtain amplitude preserved subsurface seismic reflectivities (e.g., Beylkin, 1985, Bleistein, 1987; Najmi, 1996; Bleistein et al., 2001; Schleicher et al., 2007; Koren and Ravve, 2011, and many others). These are particularly important for amplitude vs. offset/angle analyses (AVA). Geometric spreading can be obtained by solving (integrating) the dynamic ray tracing equations in ray-centered or Cartesian coordinates (e.g., Virieux et al., 1988; Červený, 2001; Iversen and de Hoop, 2018), or by directly using the second derivatives of the traveltime with respect to the source and receiver location components (e.g., Gajewski and Psencik, 1987, Červený, 2001; Vanelle and Gajewski, 2003; Xu and Tsvankin, 2006).

Our Eigenray approach proposes a finite element method, with Hermite interpolation in the intervals, for constructing stationary ray paths (Koren and Ravve, 2018). For a stationary trajectory, governed by nodal locations and orientations, the traveltime gradient vanishes, and we apply the Newton method to solve the corresponding equation set. This approach requires additional computation of the Hessian matrix representing the second derivatives of the traveltime with respect to the locations $\mathbf{x}$ and orientations $\mathbf{r}$ of the nodes. After the stationary ray path has been found, the traveltime Hessian along the stationary ray can be used to compute geometric spreading.

Geometric Spreading with Eigenray

The Eigenray method suggested in this study provides a natural way to compute geometric spreading. Along with the ray path, arc length and traveltime, the Eigenray method provides the Hessian matrix of the stationary path. The full Hessian of the traveltime represents a large, narrow-band, square matrix that includes all degrees of freedom (DoF) before imposing the boundary conditions. In this study, we suggest a way to compress this matrix into a $6 \times 6$ endpoint Hessian, whose DoF are spatial locations of the source and receiver, $x_S$ and $x_R$.

The geometric spreading is obtained directly from the Hessian matrix of the traveltime with respect to the endpoints’ location. Each node has three spatial DoF (coordinates); therefore, the endpoint Hessian is a symmetric matrix of dimension $6 \times 6$, consisting of four $3 \times 3$ blocks. The upper diagonal block is related to the source, the lower diagonal block to the receiver, and two off-diagonal ‘mixed derivative’ blocks are related to both the source and receiver. The diagonal blocks are symmetric, while the off-diagonal blocks are the transpose of each other. Even in the case of a minimum traveltime path, unlike the full Hessian that includes all nodal DoF and is subject to the boundary conditions, the endpoint Hessian is not necessarily positive definite.

Geometric spreading is computed in two stages:

- ‘Condensing’ the full traveltime Hessian matrix $\nabla_d \nabla_d t$ to the endpoint traveltime Hessian matrix $\nabla_S \nabla_R t$, where subscript d means all DoF, while S and R are related to the source and receiver, respectively. The Hessian $\nabla_d \nabla_d t$ is a narrow-band square matrix of length $6 \times 6$ number of nodes. Its bandwidth is 12 for two-node elements and 18 for three-node elements. The dimension of Hessian $\nabla_S \nabla_R t$ is $6 \times 6$ and we extract its $3 \times 3$ off-diagonal mixed block.

- Applying a conventional (well known) technique for computing the geometric spreading, given the extracted $3 \times 3$ mixed Hessian and other computed data.

We emphasize that when computing the Eigenray, the endpoint location components are fixed and not considered as DoF. The DoF are location and directional components of the internal nodes and also directional components of the end nodes. On the other hand, for the computation of the geometric spreading, only location components of the two end nodes (a total of six components) are considered independent DoF. Locations and orientations of internal nodes and orientations of end nodes are internal, dependent DoF. Given the locations of the two end nodes, all these...
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Internal DoF are fully defined by the stationary ray. There may be several stationary ray paths for the same endpoint locations (multi-pathing), each of which is treated independently. Consider an example of a ray path scheme consisting of seven nodes: two end nodes and five internal nodes, as shown in Figure 1.

![Finite element scheme of ray path](image)

Figure 1: Finite element scheme of ray path

The DoF of the full gradient and Hessian are shown in Table 1. Each small cell is a vector of length 3. External DoF, labeled $S$ and $R$, are in green, while internal DoF, labeled $A$ and $B$, are in yellow. $S$ and $R$ are locations of the source and receiver, respectively, $B$ is the ray velocity direction at the receiver, and $A$ (a longer cell) represents all other spatial and directional DoF.

<table>
<thead>
<tr>
<th>node</th>
<th>DoF</th>
<th>Symbol</th>
<th>Property</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>$x_0$</td>
<td>$S$</td>
<td>External</td>
</tr>
<tr>
<td></td>
<td>$r_0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$x_1$</td>
<td>$A$</td>
<td>Internal</td>
</tr>
<tr>
<td></td>
<td>$r_1$</td>
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</tr>
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<td>2</td>
<td>$x_2$</td>
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<td></td>
</tr>
<tr>
<td>6</td>
<td>$x_6$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$r_6$</td>
<td>$B$</td>
<td>Internal</td>
</tr>
</tbody>
</table>

Table 1: External and internal DoF.

In the Eigenray approach, the traveltine depends on all degrees of freedom,

$$ t = r(d_s, d_r, d_A, d_B) \quad (1) $$

However, if the external DoF (endpoint locations) are fixed, then the internal DoF of a paraxial ray can be computed, accounting for the vanishing traveltime gradient,

$$ d_A = d_A(d_s, d_R) \quad , \quad d_B = d_B(d_s, d_R) \quad (2) $$

The traveltime equation becomes,

$$ t = t(d_s, d_R, d_A(d_s, d_R), d_B(d_s, d_R)) = r(d_s, d_R) \quad . \quad (3) $$

That is to say that eventually, the traveltime depends on the source and receiver locations only, where the corresponding $6 \times 6$ Hessian exists and reads,

$$ \nabla_{SR} \nabla_{SR} t = \begin{bmatrix} \nabla_S \nabla_S t & \nabla_S \nabla_R t \\ \nabla_R \nabla_S t & \nabla_R \nabla_R t \end{bmatrix} \ . \quad (4) $$

The tilde is needed to distinguish these blocks from the corresponding blocks of the full Hessian. Each of the four blocks in the matrix of equation 4 is $3 \times 3$. The left lower $3 \times 3$ block consisting of the mixed derivatives (i.e., those derivatives where one coordinate belongs to the source and the other to the receiver) is needed to apply a conventional approach for computing geometric spreading (e.g., Goldin, 1986; Červený, 2001; Tsvankin and Grechka, 2011, and many others). As mentioned, the geometric spreading is a function of the compressed Hessian which, in turn, is a function of the large global Hessian related to a stationary ray path. The compression technique can be presented by a symbolic formula,

$$ \nabla_{SR} \nabla_{SR} t = \nabla_E \nabla_E t - \nabla_E \nabla t (\nabla t \nabla_E t)^{-1} \nabla t \nabla_E t \ . \quad (5) $$

where $\nabla_{SR} \nabla_{SR} t$ on the left side is the resulting compressed $6 \times 6$ endpoint Hessian, and the matrices on the right side are blocks of the global Hessian, where the boundary conditions should not be implemented.

Table 2: Blocks of traveltime Hessian related to external and internal DoF.

<table>
<thead>
<tr>
<th>SS</th>
<th>SA</th>
<th>SR</th>
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- $V_E V_E t$ is a small symmetric “external” square block, whose rows and columns are related to external DoF;
- $V_E V_I t$ is an “external/internal” rectangular block, whose rows and columns are related to external and internal DoF, respectively;
- $V_I V_E t$ is an “internal/external” rectangular block, whose rows and columns are related to internal and external DoF, respectively; we note that two rectangular blocks are transposed of each other, $V_I V_E t = (V_E V_I t)^T$;
- $V_I V_I t$ is a large symmetric “internal” square block, whose rows and columns are related to internal DoF; we note that this block is inverted in the ‘compression’ equation 5.

The global Hessian is a symmetric narrow-band matrix, but not necessarily positive definite, because the boundary conditions are not implemented. All large blocks of the global Hessian are sparse. The inverse of the band matrix $V_I V_I t$ is fully populated. There are 6 internal DoF and 6N external DoF.

Numerical Examples

We present two synthetic examples of the proposed Eigenray solutions. To date we have only obtained the computational results for isotropic velocity fields. Examples of Eigenrays in anisotropic velocity media will be presented in our future work. In the examples below, three-node finite elements are used.

Example 1: Geometric spreading for a diving wave in a medium with a constant vertical velocity gradient.

A constant velocity gradient model is defined by two parameters: the surface velocity $v_a$ and the gradient $k$.

We assume the following data: $h = 10$ km, $v_a = 2$ km/s, $k = 1$ s$^{-1}$, where $h$ is the surface offset (the chord of the circular arc). Applying exact formulae for the constant velocity gradient, we obtain the theoretical radius of the path and the horizontal slowness, $\rho = 5.38516$ km and $\rho_h = 0.185695$ s/km. We solve the problem for the stationary path with only three three-nodal elements, and obtain the nodal locations and orientations. The initial guess is the straight line on the surface, connecting the source and receiver. The finite element traveltime coincides with the theoretical value up to six digits, $t_{FE} = 3.2944642$, $t_{exact} = 3.2944623$. To establish the geometric spreading along the stationary ray path, we first compute the global Hessian of dimensions $42 \times 42$ for the seven nodes and six DoF per node. We then apply reduction formula, equation 5, to compress it into a $6 \times 6$ endpoint location Hessian, $\nabla_S R R V_{SR} t$. We remove the third row and the third column from the mixed block, and this leads to a $2 \times 2$ matrix,

$$M_{RS}^V = \begin{bmatrix} +1.60088 \times 10^{-2} & 0 \\ 0 & -1.85667 \times 10^{-2} \end{bmatrix}. \quad (6)$$

As expected, the mixed derivative matrix becomes diagonal for azimuthally isotropic media (or just isotropic as in this case), and its determinant becomes factorized. The upper left and lower right components are in-plane and out-of-plane counterparts. The take-off angles at the source and receiver points are the computed DoF,

$$\cos \theta_S = +0.927342, \cos \theta_R = -\cos \theta_S. \quad (7)$$

where $S$ and $R$ are nodes zero and six, respectively. We apply equation 5 to compute the numerical value of the geometric spreading, and we apply the theoretical formula to compute the exact value (e.g., Vanell and Gajewski, 2003),

$$L_{GS} = \sqrt{\frac{\cos \theta_{ray,R} \cos \theta_{ray,S}}{\operatorname{det} M_{RS}}} v_{ray,S} v_{ray,R} v_{phs,S} v_{phs,R}. \quad (8)$$

where $\theta_{ray}$ are the ray take-off angles, $v_{ray}$ and $v_{phs}$ are the ray and phase velocities, respectively, subscripts $S$ and $R$ are related to the source and receiver, respectively, and $M_{RS}$ is the $2 \times 2$ mixed block of the source-receiver traveltime Hessian. For azimuthally independent models, this determinant reads (Hron et al., 1986),

$$\operatorname{det} M_{RS} = \frac{1}{h} \frac{dt}{dh} \frac{d^2 t}{dt^2}, \quad \text{(9)}$$

where $h$ is the offset. The results of the computations are,

$$L_{GS}^{FE} = 53.7890$km$^2$/s, $E_R = 1.164 \times 10^{-3}, \quad \text{(10)}$$

where subscript $FE$ is related to the finite element solution, and $E_R$ is the relative error that does not exceed 0.12%. In this example, the accuracy of the geometric spreading with the finite element method (with only three elements) is very good, although the accuracy of the traveltime is better. This is not a surprise: the accuracy of the second derivatives is smaller than that of the function. In this case, the normalized geometric spreading,

$$\frac{L_{GS}}{\sigma} = 1, \quad \sigma = \int_S^R v(s) ds. \quad \text{(11)}$$

The accuracy of geometric spreading computed with paraxial rays can be slightly increased if we enforce the condition, $r_f \cdot \Delta r_f = 0$, i.e., taking into account that the
nodal directional vector is normalized, $\mathbf{r}_r \cdot \mathbf{r}_r = 1$ and cannot change its length. Thus, it can change only its direction, which means that its variation is normal to itself.

Example 2: Geometric spreading for a diving wave in a medium with a conic velocity model.

The conic velocity model (Ravve and Koren, 2007) is described by three parameters: the surface velocity $v_a$, the surface gradient $k_a$, and the asymptotic velocity $v_\infty$. We assume the same data as in the previous example for the constant velocity gradient, $h = 10\text{km}, v_a = 2\text{km/s}, k_a = 1\text{s}^{-1}$ and in addition, the asymptotic velocity, $v_\infty = 6\text{km/s}$. Applying the theory for a diving wave in the conic model, we obtain the theoretical values for the horizontal slowness $p_h$, the eccentricity $m$, the take-off angle $\theta_a$, parameter $\sigma$, and the semi-axes $A,B$ of the elliptic arc,

$$p_h = 0.259195 \text{s/km}, \quad m = 0.643016,$$

$$\theta_a = 0.544967 \text{rad}, \quad \sigma = 38.5810 \text{km}^2/\text{s},$$

$$A = 5.51257 \text{km}, \quad B = 4.22182 \text{km}.$$  

We apply the same finite element scheme and the same initial guess as in the previous example. Again, the finite element traveltime coincides with the theoretical value up to six digits, $t_{FE} = 3.6033540 \text{s}, t_{exact} = 3.6033534 \text{s}$. As in the previous case, we apply three-three-node finite elements. We obtain numerically the diagonal matrix of the mixed second derivatives of the traveltime,

$$M_{RS}^E = \begin{bmatrix} 9.33215 \times 10^{-3} & 0 \\ 0 & -3.59187 \times 10^{-2} \end{bmatrix}.$$  

Next we compute the geometric spreading analytically and numerically, to estimate the accuracy,

$$E_R = -1.215 \times 10^{-3}.$$  

As we see, the relative error is only slightly higher than 0.12%. In Figure 2 we plot the trajectories in the media with the constant gradient and with the conic velocity model. As we see, the maximum depth of the circular path exceeds that of the elliptic. The reason is that velocities for linear models are higher at the same depths. For the same reason, the traveltime of the circular path is shorter. The normalized geometric spreading in this case is $L_{GS} / \sigma = 1.42513$. Accuracies (numerical errors) of the Eigenray approach for the constant velocity gradient and the conic velocity models are summarized in Tables 3 and 4, respectively, where $\theta_a$ is the take-off angle, $z_{max}$ is the maximum penetration depth of the diving wave, $s$ is the arc length, $t$ is the traveltime, and $L_{GS}$ is the geometric spreading.

Conclusions

The Eigenray method is aimed at establishing the stationary ray path in 3D inhomogeneous general anisotropic media. With this approach, we obtain the traveltime Hessian matrix of the stationary path that allows to compute geometric spreading. The method is based on condensing the global Hessian that includes all DoF of the stationary ray path into the source-receiver Hessian with six DoF only and using its mixed block. The method does not require solution of the dynamic ray tracing equations, and may be attractive in cases where only the amplitude the Green’s function is needed and not its phase.
REFERENCES


