Slowness vector and ray velocity magnitude from ray direction in TTI media

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Summary

Two-point ray bending problems in anisotropic elastic media require finding the (non-unique) slowness vectors and the ray (group) velocity magnitudes from a given ray direction at each node of the ray trajectory. In this study, we consider polar anisotropic elastic media (transverse isotropy with tilted axis of symmetry – TTI), where the solutions for the coupled qP and qSV waves consist of a single slowness vector of a qP wave and single or triple slowness vectors of qSV waves. We provide an original and efficient solution for this challenging problem, formulated as a single polynomial equation of degree six, whose coefficients depend on the medium properties and the ray angle, where the unknown parameter is the non-unique phase angle between the slowness vectors and the tilted symmetry axis. Additionally, we provide a direct indication of the existence of a qSV triplication without the need to solve the polynomial equation. For SH waves, the solution is unique and straightforward.

Introduction

The magnitude of the ray (group) velocity v_{ray} of a given wave type vs. the ray direction $\mathbf{r}$, along with its spatial and directional derivatives, is required in many seismic-based applications in anisotropic elastic media, in particular, two-point ray-bending methods (e.g., Koren and Ravve, 2021), moveout approximations, seismic tomography, etc. This magnitude, in turn, depends on the components of the unknown slowness vector $\mathbf{p}$ to be established, $v_{\mathrm{ray}}^{-1} = \mathbf{p} \cdot \mathbf{r}$. For a given density-normalized fourth-order stiffness tensor of general anisotropy, $\tilde{\mathbf{C}}(\mathbf{x})$, a wave type and a ray direction vector $\mathbf{r}$, the resolving Hamiltonian-related equation set to compute the slowness components of $\mathbf{p}$ was suggested by Musgrave (1954) and Fedorov (1968). The Hamiltonian is based on the Christoffel equation,

$$H(\mathbf{x},\mathbf{p}) = \det(\mathbf{\Gamma} - \mathbf{I}) = 0 \, , \quad \mathbf{\Gamma}(\mathbf{x},\mathbf{p}) = \mathbf{p}\tilde{\mathbf{C}}(\mathbf{x})\mathbf{p} \, , \qquad (1)$$

where $\mathbf{\Gamma}(\mathbf{x},\mathbf{p})$ is the Christoffel 3×3 tensor and $\mathbf{I}$ is the 3×3 identity matrix. For general (triclinic) anisotropy, this equation set consists of three polynomial equations: one of degree six (associated with the vanishing Hamiltonian) and two of degree five (associated with the collinearity of the slowness-related gradient of the Hamiltonian with the ray direction). According to Bézout's theorem, the total number of roots cannot exceed the product of all the degrees of the polynomial equations, $5 \times 5 \times 6 = 150$. Grechka (2017) analyzed the algebraic complexity of the ray velocity surface and proved that the number of relevant real solutions does not exceed 19: a unique solution for the quasi-compressional wave, where the ray velocity surface is outward-convex, and up to 18 solutions for the quasi-shear waves. For polar anisotropy (TTI), Dellinger (1991) formulated the conditions for a triplication of qSV wavefronts, and Grechka (2013) demonstrated that there may be either one or three different solutions (due to triplications) for qSV slowness vectors vs. the given ray direction. According to Thomsen and Dellinger (2003), the qSV triplications in polar anisotropy occur when the ray angle moves forward, then backtracks, and then moves forward again as the phase angle uniformly increases.

In this study, we suggest a new approach to compute the slowness vectors (and the corresponding ray velocity magnitudes) for all wave modes in polar anisotropic (TTI) media, including the triplication solutions for qSV waves. This approach leads eventually to a sixth-degree polynomial equation, whose coefficients depend on the TI medium properties and the ray angle ϑ_{ray} between the ray velocity and the symmetry axis, while the roots are the solutions for the phase angles ϑ_{phs} between the slowness direction and the symmetry axis. We then show that the criterion for qSV triplications is associated with the negative discriminant of the polynomial equation and explain its computation.

Governing Equation for Phase Angle

To compute the phase angles for the coupled qP and qSV waves, we first construct the Hamiltonian defined in equation 1 with the stiffness tensor of polar anisotropy and an arbitrary orientation of the medium symmetry axis. The first equation is the vanishing Hamiltonian. The collinearity condition of the slowness-related gradient of the Hamiltonian $\nabla_{\mathbf{p}} H \equiv \partial H / \partial \mathbf{p}$ and the ray velocity direction $\mathbf{r}$ leads to a vanishing cross-product, $\nabla_{\mathbf{p}} H \times \mathbf{r} = 0$, which in turn, represents three linearly dependent Cartesian components of a vector equation. Thus, we can use only two of these components. We exploit the fact that in TTI media the three vectors - ray velocity, slowness, and axis of symmetry - are coplanar. Thus, the slowness vector $\mathbf{p}$ can be presented as a linear combination of the two known unit-length vectors: the axis of symmetry $\mathbf{k}$ and the ray direction $\mathbf{r}$, with two unknown scalar coefficients (rather than three slowness components). We introduce this linear combination in the collinearity condition, and all terms of this equation become proportional to the same nonzero vector $\mathbf{k} \times \mathbf{r}$, normal to the incidence plane (except the special case of energy propagation along the axis of symmetry). This vector can be canceled (factored out), and the equation becomes a scalar bivariate polynomial of degree three. Recall that the other scalar equation is the vanishing Hamiltonian (bivariate polynomial of degree four), and the inequality constraint about the positive projection of the slowness direction $\mathbf{n}$ on the ray direction is $\mathbf{n}\cdot\mathbf{r} = v_{\text{phs}} / v_{\text{ray}} > 0$, where v_{phs} is the phase velocity magnitude. Next, instead of searching for the two scalar coefficients of the slowness vectors, we introduce their ratio and reduce the set of two bivariate polynomial equations of degrees three and four, accompanied by the inequality constraint, to a single univariate polynomial equation of degree six,

$$d_6 c^6 + d_5 c^5 + d_4 c^4 + d_3 c^3 + d_2 c^2 + d_1 c + d_{\text{o}} = 0 \, , \quad (2)$$

where $c = \cot \vartheta_{\text{phs}}$. In terms of the Thomsen (1986) parameters, the coefficients of the polynomial read,

$$\begin{aligned}
&d_{\text{o}} = -(1+2\varepsilon)(1-f)(f+2\varepsilon)^2 m^2 \;, \\
&d_1 = +2(f+2\varepsilon)^2\left[1+\varepsilon-(1+\delta)f\right] m\sqrt{1-m^2} \;, \\
&d_2 = -4\varepsilon^2(1-f)\left[1-(1+2f)m^2\right] - 2\varepsilon f \times \\
&\left\{2(1-f)+(f+2\delta)f-\left[4-\delta(8-10f)-3(2-f)f\right]m^2\right\} - f m^2 \times \\
&\left\{(1+2\delta)f(1-2\delta-f)+\left[f+8\delta-4(2-\delta)\delta f-(1-2\delta)f^2\right]\right\} \\
&d_3 = -\left\{2(1+\varepsilon)(\varepsilon-2\delta)-\left[1+3\varepsilon-2\delta(2+\delta)\right]f+(1+\varepsilon)f^2\right\} \qquad (3) \\
&4 f m\sqrt{1-m^2} \;, \\
&d_4 = -2(1-f)f\left[2(2\delta-\varepsilon)+f\right] - f m^2 \times \\
&\left\{2\varepsilon(2-f)-\left(4\delta^2+1-f\right)f-2\delta\left[4-2(2+\varepsilon)f+f^2\right]\right\} \;, \\
&d_5 = +2f^2\left[1+\varepsilon-(1+\delta)f\right] m\sqrt{1-m^2} \;, \\
&d_6 = -(1-f)f^2\left(1-m^2\right) \;,
\end{aligned}$$

where $f = 1 - v_S^2 / v_P^2$, $m = \cos \vartheta_{ray}$, ε and δ are the Thomsen TI parameters, and v_P, v_S are the axial compressional and shear velocities, respectively. The polynomial roots are the eigenvalues of a non-symmetric companion matrix. After the phase angle is found, we compute the normalized slowness direction vector $\mathbf{n}$ from the geometric relationship,

$$\sin\vartheta_{\text{ray}}\mathbf{n} - \sin\vartheta_{\text{phs}}\mathbf{r} = \sin\left(\vartheta_{\text{ray}} - \vartheta_{\text{phs}}\right)\mathbf{k} \quad . \qquad (4)$$

Acoustic approximation

We also suggest a "light" version of equations 2 and 3 for the acoustic approximation, $f = 1$, which is a fourth-degree polynomial equation,

$$\begin{aligned}
&c^4 - \left(m / \sqrt{1-m^2}\right)(1+2\delta)c^3 - 4(\varepsilon-\delta)c^2 \\
&-(1+2\delta)\left(\sqrt{1-m^2} / m\right)c + (1+2\varepsilon)^2 = 0 \;.
\end{aligned} \qquad (5)$$

Slowness Magnitude

We then apply the Christoffel equation to establish the slowness magnitude p for qP and qSV waves, using the quadratic equation for the slowness squared,

$$\begin{aligned}
&\left[1 - f + 2(\varepsilon - f\delta)\sin^2\vartheta_{\text{phs}} - 2f(\varepsilon-\delta)\sin^4\vartheta_{\text{phs}}\right] v_P^4 p^4 \\
&\qquad -\left(2 - f + 2\varepsilon\sin^2\vartheta_{\text{phs}}\right) v_P^2 p^2 + 1 = 0.
\end{aligned} \qquad (6)$$

Triplication Criterion in Ray-Angle Domain

Equation 2 has at least two and at most four real roots, where one root is compressional, and the other one or three are related to qSV waves. The triplication occurs in the case of a negative discriminant of the six-order polynomial equation, which in turn, represents the determinant of the Sylvester (1851) square matrix $\mathbf{S}$ of dimension 11,

$$\begin{bmatrix}
d_6 & d_5 & d_4 & d_3 & d_2 & d_1 & d_{\text{o}} & 0 & 0 & 0 & 0 \\
0 & d_6 & d_5 & d_4 & d_3 & d_2 & d_1 & d_{\text{o}} & 0 & 0 & 0 \\
0 & 0 & d_6 & d_5 & d_4 & d_3 & d_2 & d_1 & d_{\text{o}} & 0 & 0 \\
0 & 0 & 0 & d_6 & d_5 & d_4 & d_3 & d_2 & d_1 & d_{\text{o}} & 0 \\
0 & 0 & 0 & 0 & d_6 & d_5 & d_4 & d_3 & d_2 & d_1 & d_{\text{o}} \\
6d_6 & 5d_5 & 4d_4 & 3d_3 & 2d_2 & d_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 6d_6 & 5d_5 & 4d_4 & 3d_3 & 2d_2 & d_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 6d_6 & 5d_5 & 4d_4 & 3d_3 & 2d_2 & d_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 6d_6 & 5d_5 & 4d_4 & 3d_3 & 2d_2 & d_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 6d_6 & 5d_5 & 4d_4 & 3d_3 & 2d_2 & d_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 6d_6 & 5d_5 & 4d_4 & 3d_3 & 2d_2 & d_1
\end{bmatrix} \qquad (7)$$

where d_i are the coefficients listed in equation 3.

SH Slowness Vector

For SH waves, the solution is unique and reads,

$$\mathbf{p} = \frac{2\gamma m \mathbf{k} + \mathbf{r}}{v_P\sqrt{1-f}\sqrt{1+2\gamma}\sqrt{1+2\gamma m^2}} \quad , \qquad (8)$$

where γ is the corresponding Thomsen parameter. The magnitudes of the phase and ray velocities are,

$$\frac{v_{\text{phs}}}{v_P} = \frac{\sqrt{1-f}\sqrt{1+2\gamma}\sqrt{1+2\gamma m^2}}{\sqrt{1+4\gamma(1+\gamma)m^2}}, \frac{v_{\text{ray}}}{v_P} = \frac{\sqrt{1-f}\sqrt{1+2\gamma}}{\sqrt{1+2\gamma m^2}} \quad (9)$$

Particular Cases of Ray Angle

For zero ray angles (axial propagation) and 90^{o} (propagation in the normal plane), the phase angles of the compressional wave and the non-triplicated qSV wave coincide with the ray angles. Additionally, equation 2 reduces to a bi-quadratic equation, which may yield two symmetric triplicated SV shear solutions.

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Separation between Compressional and Shear Roots

Solving polynomial equation 2, we obtain two or four different real solutions for the phase angles $\vartheta_{\text{phs}} = \operatorname{arccot} c$, and it is not immediately clear which of them is related to the qP wave. To understand this, we first compute the ray angles for all the solutions. Assume, for example that the triplication exists, and we obtained four phase angle solutions. Considering each of them as a candidate for both qP and qSV waves, we compute eight ray angles: Four of them (one qP and three qSV) will be identical, coinciding with the given (input) ray angle, $\vartheta_{\text{ray}} = \arccos m$. The other four computed ray angles differ from the given value and belong to irrelevant solutions. To compute the ray angle we first compute the phase velocity magnitude v_{phs} and its derivative v'_{phs} with respect to the phase angle ϑ_{phs},

$$\frac{v_{\text{phs}}^2\left(\vartheta_{\text{phs}}\right)}{v_P^2} = 1 - \frac{f}{2} + \varepsilon \sin^2 \vartheta_{\text{phs}} \pm \frac{1}{2}\sqrt{\left(f + 2\varepsilon \sin^2 \vartheta_{\text{phs}}\right)^2 - 2f\left(\varepsilon - \delta\right)\sin^2 2\vartheta_{\text{phs}}} \quad , \qquad (10)$$

with the upper sign for qP-waves and the lower for qSV-waves (Tsvankin, 2001). Note that equation 10 follows from equation 6. Finally, the ray angle is,

$$\vartheta_{\text{ray}} = \vartheta_{\text{phs}} + \arctan\left(v'_{\text{phs}} / v_{\text{phs}}\right) \quad . \qquad (11)$$

Numerical Example

Consider a polar anisotropic medium with the properties suggested by Grechka (2013), $v_P = 3\ \text{km/s}$, $f = 0.75$, $\delta = 0.3, \varepsilon = -0.15$. Note that although the case where $\varepsilon < 0$ is not typical for consolidated shale/sand rocks, it is a common characteristic, for example, for shallow unconsolidated sand rocks (e.g., Bachrach et al. 2000). We choose these parameters in order to reproduce the triplication phenomena of qSV waves. In Figures 1a and 1b, we plot the magnitudes of the phase (blue line) and the ray (orange line) velocities of the qP and SH waves, respectively. In Figures 1c and 1d we plot the magnitudes of the phase and ray velocities, respectively, of the triplicated qSV wave. As noted by Grechka (2013), in this specific medium the triplications occur in the proximity of the axis of symmetry and in the proximity of the isotropic plane (normal to the axis). As can be seen in Figures 1c and 1d, at ray angles zero and 90^{o}, the expected magnitudes of both qSV phase and ray velocities are present. The ray and phase velocities of this wave are shown by blue lines. However, due to the triplication, there are also two other qSV waves, with much lower phase and ray velocities shown by orange and green lines, respectively. The phase and ray velocities of the "orange" and "green" qSV waves coincide at the ends of the interval (ray angles zero and 90^{o}) but differ elsewhere. In the proximity of 45^{o}, there is no triplication: only the "green" qSV wave exists. In Figure 1e, we plot the signed lead/lag angle between the ray and phase velocity directions of the qP (blue line) and SH (orange line) waves, respectively. For the given model, this angle is always negative or zero (i.e., the lag) for the SH wave and alternates in sign for the qP wave. In Figure 1f, we plot the lead/lag for the triplicated qSV wave. At both ends of the interval, the lag of the "blue" qSV wave is zero. This is a "regular" qSV wave, with both ray and phase velocity directions along the medium symmetry axis, or both in the isotropic plane (with identical azimuths). This is not so for the "orange" and "green" waves. In the isotropic plane (ray angle 90^{o}), the "orange" and "green" qSV waves have symmetric lag -39.39^{o} and lead $+39.39^{\text{o}}$, respectively. For the axial direction of the ray velocity, the lag is -44.08^{o} for both the "orange" and "green" waves. The absolute value of this angle defines the deviation of the phase velocity direction from the medium axis of symmetry (which is extremely large in this example), but the phase azimuth remains undetermined. In Figure 1g we plot the discriminant of the governing sixth-order polynomial equation vs. the ray angle. Within the range $30^{\text{o}} \lesssim \vartheta_{\text{ray}} \lesssim 55^{\text{o}}$, the discriminant is close to zero, and we zoom in to show its sign in Figure 1h.

Conclusions

Considering elastic media characterized by polar anisotropy (TTI), we suggest a new method for computing the slowness vectors and corresponding ray velocity magnitudes, given the medium properties, the orientation of the symmetry axis, and the ray velocity direction. This kind of inversion is inherent in the solution of two-point ray bending problems. For the coupled qP and qSV waves, we show that the solutions can be obtained by finding the roots of a single univariate polynomial equation of degree six, whose coefficients depend on the medium properties and the ray angle, where the unknown parameter is the non-unique phase angle between the slowness vector and the symmetry axis. The six roots include a unique solution for qP waves and one or three solutions for qSV waves (depending on the existing or non-existing triplication). In addition, there are one or two pairs of complex-conjugate roots. One pair of complex-conjugate roots always exists and has a "cuspoidal" nature. The second pair exists only in the absence of a qSV triplication. We also provide a direct indication for the existence of a qSV triplication without the need to solve the polynomial equation.

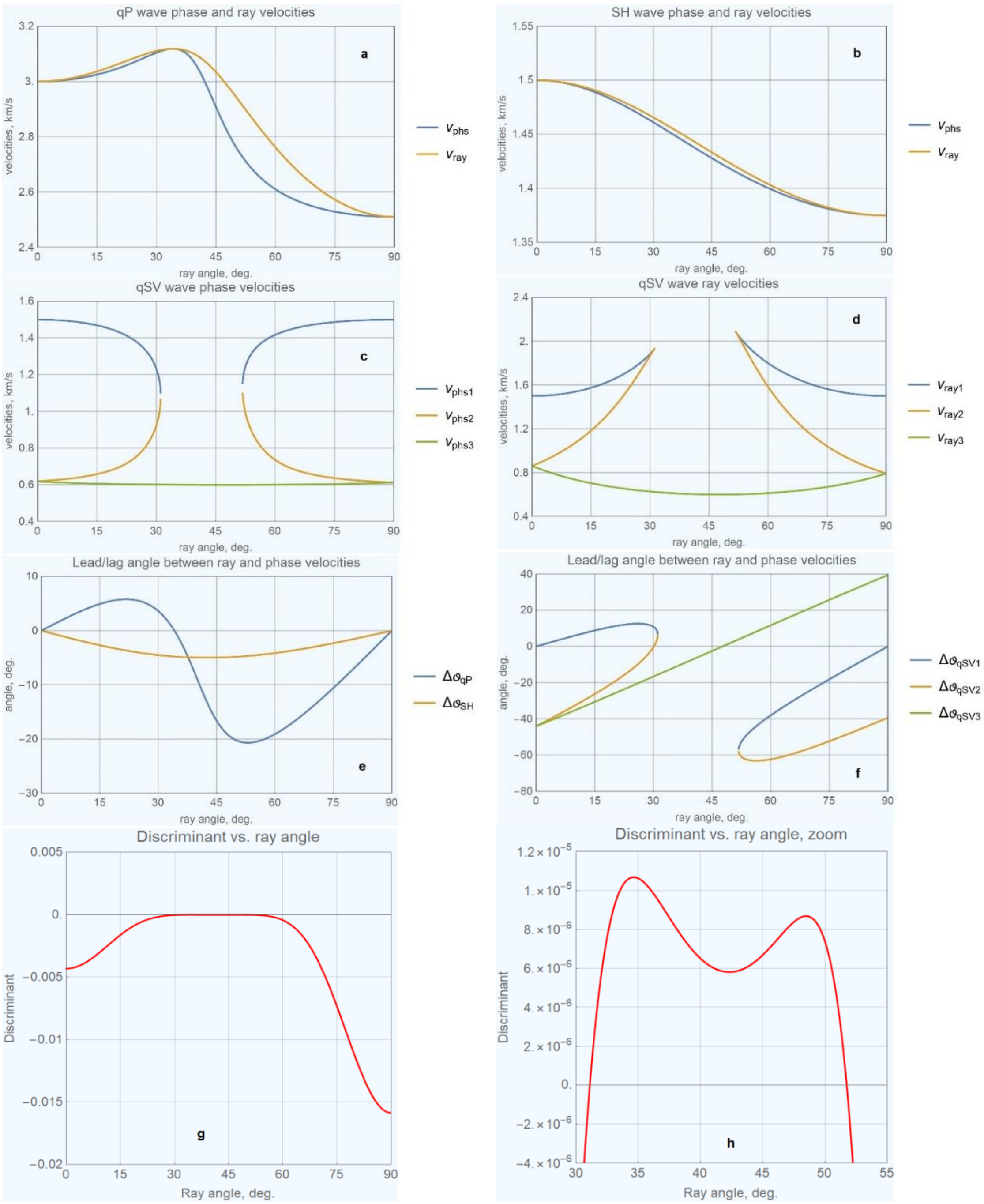

Figure 1: Characteristics of a TTI model vs. the ray angle between the axis of symmetry and the ray direction: a) phase and ray velocity magnitudes for the qP wave; b) phase and ray velocity magnitudes for the SH wave; c) phase velocity magnitudes for the triplicated qSV wave; d) ray velocity magnitudes for the triplicated qSV wave; e) lead/lag angles between the ray and phase velocity directions vs. ray angle for qP and SH waves; f) lead/lag angles between the ray and phase velocity directions vs. ray angle for the triplicated qSV wave; g) discriminant of the governing polynomial vs. the ray angle; h) discriminant of the governing polynomial, zoom in.